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SECOND ORDER FAMILIES OF SPECIAL LAGRANGIAN SUBMANIFOLDS IN $\mathbb{C}^4$

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Abstract

This paper extends to dimension 4 the results in the article "Second order families of special Lagrangian 3-folds" by Robert Bryant. We consider the problem of classifying the special Lagrangian 4-folds in $\mathbb{C}^4$ whose fundamental cubic at each point has a nontrivial stabilizer in SO(4). Points on special Lagrangian 4-folds where the SO(4)-stabilizer is nontrivial are the analogs of the umbilical points in the classical theory of surfaces. In proving existence for the families of special Lagrangian 4-folds, we used the method of exterior differential systems in Cartan-Kähler theory. This method is guaranteed to tell us whether there are any families of special Lagrangian submanifolds with a certain stabilizer type, but does not give us an explicit description of the submanifolds. To derive an explicit description, we looked at foliations by submanifolds and at other geometric particularities. In this manner, we settled many of the cases and described the families of special Lagrangian submanifolds in an explicit way.

1. Introduction

The complex space $\mathbb{C}^m$ is endowed with a Kähler form

$$\omega = \frac{i}{2}(dz_1 \wedge d\bar{z}_1 + dz_2 \wedge d\bar{z}_2 + \cdots + dz_m \wedge d\bar{z}_m)$$

and a volume form

$$\Omega = dz_1 \wedge dz_2 \wedge \cdots \wedge dz_m,$$

where $(z_1, z_2, \ldots, z_m)$ are the coordinates on $\mathbb{C}^m$. A *special Lagrangian submanifold* in $\mathbb{C}^m$ is an m-dimensional real submanifold on which the forms ω and $\operatorname{Im}\Omega$ restrict to 0.

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The study of special Lagrangian (SL) submanifolds started with Harvey and Lawson in their paper [12] on calibrated geometries. They constructed many interesting examples of SL m-folds in $\mathbb{C}^m$ and proved local existence theorems. Since then, many other examples have been constructed using a variety of techniques. To give some examples, Dominic Joyce used the method of ruled submanifolds, integrable systems and evolution of quadrics in [8], [9], [10] to construct explicit examples of special Lagrangian m-folds in $\mathbb{C}^m$, Mark Haskins exhibited examples of special Lagrangian cones in $\mathbb{C}^3$ [4], Richard Schoen and Jon Wolfson used the variational approach for some of their constructions in Calabi-Yau manifolds in [15], etc.

Special Lagrangian geometry received reinforced attention in 1996 when Strominger, Yau and Zaslow formulated what is today known as the SYZ conjecture [17]. This conjecture reveals the role of the special Lagrangian geometry in mirror symmetry, a mysterious relationship between pairs of Calabi-Yau 3-folds, coming from string theory. In this larger context, a lot of research is going on nowadays to find examples of special Lagrangian submanifolds. This would help in understanding what kind of singularities a special Lagrangian submanifold in a Calabi-Yau can have, classifying them and maybe ultimately resolving the SYZ conjecture.

While, from the string theory point of view, the most interesting case to study is the special Lagrangian 3-folds of a Calabi-Yau 3-fold, higher dimensional cases are also important for the understanding of the general theory of SL submanifolds in Calabi-Yau m-folds.

The idea in this research, initiated by Robert Bryant in his paper [1], is to classify families of SL submanifolds that are characterized by invariant, geometric conditions. When the ambient space is flat, the second fundamental form is the lowest order invariant of a SL submanifold, so we would like to study the second order families of SL m-folds in $\mathbb{C}^m$, that is the families of SL m-folds in $\mathbb{C}^m$ whose second fundamental form satisfies a set of pointwise conditions.

The second fundamental form of a special Lagrangian submanifold in $\mathbb{C}^m$ has a natural interpretation as a traceless cubic form on the submanifold, called the *fundamental cubic*. The stabilizer at a generic point of the fundamental cubic of a generic SL m-fold is trivial. For comparison, in the case of a hypersurface in $\mathbb{R}^{m+1}$, the stabilizer of the second fundamental form in $\mathrm{SO}(m)$ is always nontrivial and is larger than the minimum possible stabilizer exactly at the umbilical points of the hypersurface. For this reason, the points on SL m-folds where the

SO(m)-stabilizer is nontrivial are the analogs of the umbilical points in the classical theory of surfaces.

In his article [1], Robert Bryant considered the 'umbilical' case and completely classified the SL submanifolds of $\mathbb{C}^3$ whose fundamental cubic has nontrivial SO(3)-stabilizer at a generic point. He found that the only SL 3-folds whose fundamental cubic has a nontrivial stabilizer at a generic point are the 3-planes, with stabilizer SO(3), the Harvey and Lawson examples, with stabilizer SO(2), the austere SL 3-folds, with stabilizer S_3, the asymptotically conical SL 3-folds, with stabilizer $\mathbb{Z}_3$ and the Lawlor-Harvey-Joyce examples, with stabilizer $\mathbb{Z}_2$.

This present work extends these results to dimension $m = 4$, namely tries to classify the special Lagrangian 4-folds in $\mathbb{C}^4$ whose fundamental cubic at a generic point has nontrivial SO(4)-stabilizer.

The possible stabilizer of a traceless cubic can be a continuous, meaning a positive dimensional, or a discrete subgroup of SO(4). In Chapter 3.2, we consider the case when the stabilizer is continuous. It turns out that there are four cases when there are nontrivial special Lagrangian 4-folds with continuous stabilizer type:

(a) When the fundamental cubic has stabilizer SO(3) we obtain the Harvey and Lawson examples which appeared also in dimension 3: $L_c = \{(s+it)\mathbf{u} \mid \mathbf{u} \in S^3 \subset \mathbb{R}^4,\ \mathrm{Im}(s+it)^4 = c\}$, where c is any real constant.

(b) When the stabilizer is $\mathrm{SO}(2) \ltimes S_3$, we obtain special Lagrangian submanifolds as products of the form $L = \mathbb{R}^2 \times \Sigma$, where $\Sigma \subset \mathbb{C}^2$ is a complex curve.

(c) When the stabilizer is SO(2), we obtain the SO(3)-invariant special Lagrangian 4-folds.

(d) When the stabilizer is an O(2), we obtain a two parameter family of solutions which we have not been able to integrate completely yet.

In Chapter 3.3, we consider the case when the stabilizer of the fundamental cubic is a discrete subgroup of SO(4). In Chapter 3.3.1, we classify the SL 4-folds with polyhedral stabilizer type. It turns out that the polyhedral subgroups of SO(4) that stabilize a traceless cubic in 4 variables are the tetrahedral subgroup $\mathbb{T}$, the irreducibly acting octahedral subgroup $\mathbb{O}^+$ and the irreducibly acting icosahedral subgroup $\mathbb{I}^+$. We show that the special Lagrangian 4-folds whose stabilizer of its fundamental cubic is isomorphic to the tetrahedral subgroup are the Harvey-Lawson examples invariant under a torus action, the ones whose stabilizer at a generic point is isomorphic to $\mathbb{O}^+$ are the cones on flat 3-dimensional tori in the 7-sphere and that there are no nontrivial special Lagrangian 4-folds whose stabilizer of its fundamental cubic at a generic point is isomorphic to $\mathbb{I}^+$.

Using the classification of the discrete subgroups of SO(4) from Chapter 3.1, it remains to analyze the cases when the stabilizer of the traceless cubic is a cyclic or a dihedral subgroup of SO(4). We show that the discrete stabilizer can only have elements of order less or equal to 6. Further, we show that if the stabilizer is discrete and contains an element of order 6, 5 or 4, then there are no special Lagrangian 4-folds in $\mathbb{C}^4$ with a cyclic or dihedral stabilizer type.

When the stabilizer contains an element of order 3, there are two inequivalent orbits in the space of fixed traceless cubics that have to be considered. In the first case of discrete stabilizer type at least a $\mathbb{Z}_3$, the special Lagrangian 4-folds whose cubic stabilizer at a generic point is isomorphic to D_3, the dihedral group in 3 elements, turn out to be asymptotically conical. The SL 4-folds with stabilizer type an order 18 normal subgroup of $D_3 \times D_3$ turn out to be products of holomorphic curves. When the stabilizer type is exactly a $\mathbb{Z}_3$, we were able to show that there is an infinite parameter family of solutions that depends on 4 functions of 1 variable, foliated by minimal Legendrian surfaces and by holomorphic curves, but could not finalize the analysis and describe this family completely. In the second case of discrete stabilizer type at least $\mathbb{Z}_3$, we found a large family of SL 4-folds defined by holomorphic differential equations, a family that did not appear in dimension 3.

The general case of discrete stabilizer type at least a $\mathbb{Z}_2$ is the most complicated case, since the general traceless cubic has a large number of parameters, and was not considered in this work.

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2. Special Lagrangian geometry in Calabi-Yau manifolds

2.1 Special Lagrangian submanifolds

We begin with the definition of a Calabi-Yau manifold.

Definition 2.1. A Calabi-Yau m-fold (M, J, g) is a compact, complex m-dimensional manifold (M, J) with trivial canonical bundle K_M and Ricci-flat Kähler metric g.

Because the canonical bundle K_M is trivial, there is a nonzero holomorphic section Ω of K_M. Since the metric g is Ricci-flat, Ω is a parallel tensor with respect to the Levi-Civita connection ∇^g [6]. By rescaling Ω, we can take it to be the holomorphic $(m, 0)$-form that satisfies:

$$\frac{\omega^m}{m!} = (-1)^{\frac{m(m-1)}{2}} \left(\frac{i}{2}\right)^m \Omega \wedge \overline{\Omega}, \tag{2.1}$$

where ω is the Kähler form of g. The form Ω is called the *holomorphic volume form* of the Calabi-Yau manifold M.

The special Lagrangian submanifolds were introduced by Harvey and Lawson in their paper [12] using calibrations. They are defined in the general setting of a Calabi-Yau manifold and are a special class of minimal submanifolds.

Definition 2.2. Let (M, J, g, Ω) be a Calabi-Yau m-fold and $L \subset M$ a real m-dimensional submanifold of M. Then L is called a *special Lagrangian submanifold* of M if $\omega\,|_L \equiv 0$ and $\operatorname{Im}\Omega\,|_L \equiv 0$.

More generally, L is said to be a special Lagrangian submanifold with phase $e^{i\theta}$ if $\omega\,|_L \equiv 0$ and $\operatorname{Im}(e^{i\theta}\Omega)\,|_L \equiv 0$.

As an example, we can see that $\mathbb{R}^m \subset \mathbb{C}^m$ is a special Lagrangian subspace. $\mathbb{C}^m$ is endowed with the standard Calabi-Yau structure defined by

$$g_0 = dz_1 \circ d\bar{z}_1 + \cdots + dz_m \circ d\bar{z}_m \tag{2.2}$$

$$\omega_0 = \frac{i}{2}(dz_1 \wedge d\bar{z}_1 + \cdots + dz_m \wedge d\bar{z}_m) \tag{2.3}$$

$$\Omega_0 = dz_1 \wedge \cdots \wedge dz_m \tag{2.4}$$

where g_0 is the Kähler metric on $\mathbb{C}^m$, ω_0 the Kähler form and Ω_0 the holomorphic volume form on $\mathbb{C}^m$. An m-dimensional submanifold L of M is called *Lagrangian* if $\omega\,|_L = 0$. So, the special Lagrangian submanifolds of M are the Lagrangian submanifolds with the extra condition $\operatorname{Im}\Omega\,|_L = 0$, which is exactly the reason for their name.

There are some important results on SL submanifolds which we will briefly recall here.

a. Deformations. R. McLean [13] studied the moduli space of compact special Lagrangian deformations and showed that it has the following description:

Theorem 2.3 (McLean)**.** *Let (M, J, g, Ω) be a Calabi-Yau m-fold and $L \subset M$ a m-dimensional compact SL submanifold. Then the moduli space $\mathcal{M}_L$ of special Lagrangian deformations of L is a smooth manifold of dimension $b^1(L)$, the first Betti number.*

b. Local existence. Harvey and Lawson [12] proved local existence only for SL-submanifolds in $\mathbb{C}^m$, but their result extends to show that if (M, J, g, Ω) is a Calabi-Yau m-fold and $N \subset M$ a real analytic submanifold of dimension $m-1$ such that $i^*(\omega) = 0$, then N lies in a unique irreducible SL submanifold $L \subset M$. Here $i\colon N \to M$ is the inclusion map.

This result shows that there are many special Lagrangian submanifolds locally.

c. Minimizing property. A closed special Lagrangian submanifold is volume-minimizing in its homology class and therefore it is a minimal submanifold. We remark that a minimal submanifold, i.e., a submanifold with constant mean curvature 0, is not necessarily volume-minimizing amongst homologous submanifolds. For example the equator of a 2-dimensional sphere is minimal, but does not minimize length amongst lines of latitude.

2.2 Structure equations

a. The coframe bundle. Let (M, J, g, Ω) be a Calabi-Yau m-fold and let $\mathbb{C}^m \cong \mathbb{R}^{2m}$ have complex coordinates $(z_1, z_2, \dots, z_m)$ and complex structure I. The standard Calabi-Yau structure on $\mathbb{C}^m$ is given by the relations (2.2), (2.3) and (2.4). Let $\pi\colon P \to M$ denote the bundle of $\mathbb{C}^m$-valued Calabi-Yau coframes, i.e., an element of $P_x = \pi^{-1}(x)$ is a complex linear vector space isomorphism $u\colon T_xM \to \mathbb{C}^m$ that satisfies $\omega_x = u^*(\omega_0)$ and $\Omega_x = u^*(\Omega_0)$. Then $\pi\colon P \to M$ is a principal right $\mathrm{SU}(m)$-bundle over M and the right action is given by $R_a(u) = a^{-1} \circ u$ for $a \in \mathrm{SU}(m)$. P_x is the fiber at x of the Calabi-Yau coframe bundle P.

The canonical form ξ is defined on the Calabi-Yau coframe bundle P by

$$\xi_u = u \circ (d\pi)_u\colon T_uP \to \mathbb{C}^m \quad \text{for } u \in P$$

where $(d\pi)_u\colon T_uP \to T_{\pi(u)}M$ is the differential of π at u. The 1-form ξ is $\mathbb{C}^m$-valued and we denote its components by ξ_i, $i = 1 \dots m$. Then, on the bundle P the following equations hold:

$$\pi^*(\omega) = \frac{i}{2}(\xi_1 \wedge \bar{\xi}_1 + \cdots + \xi_m \wedge \bar{\xi}_m) \quad \text{and} \quad \pi^*(\Omega) = \xi_1 \wedge \cdots \wedge \xi_m. \tag{2.5}$$

By regarding the forms on M embedded into the forms on P via the pullback, we can ignore π^* in the above equations.

We define also the functions $e_i \colon P \to TM$ such that $\xi_i(e_j) = \delta_{ij}$. So, if $v \in T_uP$ then: $(d\pi)_u(v) = e_i(u)\xi_i(v)$. Cartan's first structure equation:

$$d\xi_i = -\psi_{i\bar{\jmath}} \wedge \xi_j \tag{2.6}$$

defines $(\psi_{i\bar{\jmath}}) = \psi$, the $\mathfrak{su}(m)$-valued 1-form on P called the *connection form*. In the flat case $M = \mathbb{C}^m$, Cartan's second structure equation satisfied by the connection form ψ is:

$$d\psi = -\psi \wedge \psi. \tag{2.7}$$

b. Special Lagrangian submanifolds in $\mathbb{C}^m$. In this paper we are interested in special Lagrangian submanifolds of $\mathbb{C}^m$, therefore we are considering only the flat case from now on. When $M = \mathbb{C}^m$ with the standard Calabi-Yau structure $(\mathbb{C}^m, J, g_0, \Omega_0)$, we denote the Calabi-Yau coframe bundle by $\mathrm{x} \colon P \to \mathbb{C}^m$ and regard the functions e_i as vector-valued functions on $P \cong \mathbb{C}^m \times \mathrm{SU}(m)$. Then the relations:

$$d\mathrm{x} = e_i \xi_i \tag{2.8}$$

$$de_i = e_j \psi_{j\bar{\imath}} \tag{2.9}$$

give the 1-forms $\{\xi_i, \psi_{i\bar{\jmath}}\}$ which form a basis for the space of 1-forms on the frame bundle P.

To study the SL submanifolds of $\mathbb{C}^m$, we separate the two structure equations (2.6) and (2.7) into real and imaginary parts. We set $\xi_i = \omega_i + i\eta_i$ and $\psi_{i\bar{\jmath}} = \alpha_{ij} + i\beta_{ij}$. The first structure equation (2.6) becomes

$$d\omega_i = -\alpha_{ij} \wedge \omega_j + \beta_{ij} \wedge \eta_j \quad \text{and} \quad d\eta_i = -\beta_{ij} \wedge \omega_j - \alpha_{ij} \wedge \eta_j \tag{2.10}$$

where we used Einstein's convention to sum over repeated indices. Since ψ is skew-hermitian with trace 0, it follows that $\alpha = (\alpha_{ij})$ is skew-symmetric and $\beta = (\beta_{ij})$ is symmetric with vanishing trace.

When split into its real and imaginary parts, the second structure equation (2.7) becomes:

$$d\alpha_{ij} = -\alpha_{ik} \wedge \alpha_{kj} + \beta_{ik} \wedge \beta_{kj} \tag{2.11}$$

$$d\beta_{ij} = -\beta_{ik} \wedge \alpha_{kj} - \alpha_{ik} \wedge \beta_{kj}. \tag{2.12}$$

Let $L \subset M$ be a special Lagrangian submanifold. We are going to consider the bundle P_L of L-adapted coframes over L. This is defined as follows: Let $x \in L$. A Calabi-Yau coframe at x, $u\colon T_xM \to \mathbb{C}^m$ is said to be *L-adapted* if $u(T_xL) = \mathbb{R}^m \subset \mathbb{C}^m$ and $u\colon T_xL \to \mathbb{R}^m$ preserves orientation. The space of L-adapted coframes forms a principal right $\mathrm{SO}(m)$-subbundle $P_L \subset \pi^{-1}(L) \subset P$ over L. Now, because u takes a tangent plane to L in M into a real one, ξ is $\mathbb{R}^m$-valued on P_L and so $\eta_i = 0$ holds on P_L. By the structure equation (2.10), we get that:

$$d\omega_i = -\alpha_{ij} \wedge \omega_j \quad \text{and} \quad \beta_{ij} \wedge \omega_j = 0 \quad \text{on } P_L.$$

Since $\omega_1, \dots, \omega_m$ are linearly independent forms and $\beta_{ij} \wedge \omega_j = 0$, Cartan's Lemma implies that $\beta_{ij} = h_{ijk}\omega_k$ where $h_{ijk} = h_{ikj}$. Since β_{ij} is symmetric, $h_{ijk} = h_{jik}$ also holds and so h_{ijk} are fully symmetric functions on the bundle P_L.

c. The fundamental cubic. Let $L \subset M = \mathbb{C}^m$ be a SL submanifold and let $\nu \to L$ be the normal bundle of L in M, such that $TM\,|_L = TL \oplus \nu$. The second fundamental form of L is a quadratic form with values in the normal bundle ν and it can be interpreted as a traceless symmetric cubic form in the following way: The second fundamental form B of L in M can be written as $B = Je_i \otimes h_{ijk}\omega_j\omega_k$, where h_{ijk} are the fully symmetric functions determined by β_{ij} as described above. All the information of the second fundamental form is contained in the symmetric cubic form $C = h_{ijk}\omega_i\omega_j\omega_k$ which is called the *fundamental cubic* of the special Lagrangian submanifold L. We note that this cubic is traceless with respect to the induced metric on L, $g = \omega_1^2 + \cdots + \omega_m^2$, since:

$$\mathrm{tr}_g C = h_{iik}\omega_k = \beta_{ii} = 0.$$

The following result tells us that the necessary and sufficient conditions for the existence of a special Lagrangian in $\mathbb{C}^m$ with a given metric and a given fundamental cubic are the Gauss and Codazzi-type equations (2.11) and (2.12).

Let $(L, g) \subset \mathbb{C}^m$ be a simply connected Riemannian manifold of dimension m and C a symmetric cubic which is traceless with respect to g. Choose a g-orthonormal coframing $\omega = (\omega_i)$ on an open neighborhood $U \subset L$ and define $\eta_i = 0$. Now, let $\alpha_{ij} = -\alpha_{ji}$ be the unique 1-forms on U s.t. $d\omega_i = -\alpha_{ij} \wedge \omega_j$. Write the cubic as $C = h_{ijk}\omega_i\omega_j\omega_k$ and set $\beta_{ij} = h_{ijk}\omega_k$.

Theorem 2.4 (see [1]). *Suppose that the forms β_{ij} determined by C together with the forms α_{ij} determined by (ω_i) satisfy the Gauss and*

Codazzi equations (2.11), (2.12). *Then there is an isometric immersion of* (L, g) *into* $\mathbb{C}^m$ *as a special Lagrangian submanifold inducing* C *as its fundamental cubic. Moreover, this isometric immersion is unique up to rigid motions.*

3. Second order families

3.1 Discrete subgroups of SO(4)

As we have seen in Section 2.2.c, the second fundamental form C of a special Lagrangian submanifold $L \subset \mathbb{C}^m$ can be regarded as a symmetric cubic form in n variables $x_1, x_2, \ldots, x_n$, with vanishing trace with respect to the induced metric g. It is easy to see that the symmetric cubic is traceless if and only if it is a *harmonic* cubic, i.e., $\Delta C = 0$, where $\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$. Therefore, the fundamental cubic of a special Lagrangian submanifold in $\mathbb{C}^m$ belongs to the space $\mathcal{H}_3(\mathbb{R}^4)$ of harmonic cubics in 4 variables. This space is an irreducible SO(4)-module of dimension 16 and the action is given by

$$(A \cdot P)x = P(xA) \tag{3.13}$$

where

$$A = (a_{ij}) \in \mathrm{SO}(4), \quad P(x) \in \mathcal{H}_3(\mathbb{R}^4), \quad x = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4$$

and

$$(xA)_i = x_j a_{ji}$$

is given by usual matrix multiplication.

We want to study the families of special Lagrangian 4-folds in $\mathbb{C}^4$ whose fundamental cubic at a generic point has nontrivial SO(4)-stabilizer. The stabilizer G of a polynomial $P(x) \in \mathcal{H}_3(\mathbb{R}^4)$ is defined as

$$G = \{A \in \mathrm{SO}(4) |\ (A \cdot P)(x) = P(x), \text{ for any } x \in \mathbb{R}^4\}.$$

The stabilizer can be either a positive dimensional subgroup of SO(4) or else a discrete subgroup of SO(4). In our analysis, we need to know which are the discrete subgroups of SO(4) that can stabilize a harmonic cubic in 4 variables.

We start by listing the discrete subgroups of SO(4) not containing the central rotation $-I_4$, since a subgroup of SO(4) that contains $-I_4$

cannot stabilize any nontrivial cubic polynomial. For a complete proof of the classification of the discrete subgroups of SO(4) the reader might want to consult [11].

In the study of the discrete subgroups of SO(4), we are going to use the quaternionic field $\mathbb{H}$. Let $\mathbb{E}_4$ be the Euclidean 4-dimensional space and let $\{\mathbf{1}, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$ be an orthonormal basis. We define a multiplication of elements of $\mathbb{E}_4$ by the well-known rules: $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -\mathbf{1}$, $\mathbf{ij} = \mathbf{k}$, $\mathbf{jk} = \mathbf{i}$, $\mathbf{ki} = \mathbf{j}$. The elements of $\mathbb{E}_4$ form the non-commutative field of quaternions $\mathbb{H}$. We will denote a quaternion by the ordered set (w, x, y, z) or by $w + x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. For a quaternion $q = w + x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ we define the conjugate $\bar{q} = w - x\mathbf{i} - y\mathbf{j} - z\mathbf{k}$ and the modulus of q as $|q| = (q\bar{q})^{\frac{1}{2}}$. If $|q| = 1$, we call q a unit quaternion and $U = \{q \in \mathbb{H} \mid |q| = 1\}$ is a multiplicative group called the group of unit quaternions.

For any $q \in \mathbb{E}_4$, we define the *right multiplication map* $\rho_q \colon \mathbb{E}_4 \to \mathbb{E}_4$ by $\rho_q(x) = x\bar{q}$ and the *left multiplication map* $\lambda_q \colon \mathbb{E}_4 \to \mathbb{E}_4$ by $\lambda_q(x) = qx$. If $u \in U$, both ρ_u and λ_u are seen to be in SO(4) and they are called the right rotation and the left rotation, respectively. The right rotations $\{\rho_u : u \in U\}$ form a group U_+ and the left rotations $\{\lambda_u : u \in U\}$ form a group U_- and both U_+ and U_- are subgroups of SO(4), isomorphic to the unit quaternions group U. These are different subgroups of SO(4) and we notice that $U_+ \cap U_- = \{\pm 1\}$.

Consider now the homomorphism $\Phi \colon U \times U \to \mathrm{SO}(4)$ with

$$\Phi(u_1, u_2)(x) = u_1 x \bar{u}_2 = \lambda_{u_1} \rho_{u_2}(x).$$

It is well-known that Φ is surjective and its kernel is the 2-element group $\{(1,1), (-1,-1)\} \cong \mathbb{Z}_2$. So, $U \times U/_{\{(1,1),(-1,-1)\}} \cong \mathrm{SO}(4)$ and to study the subgroups of SO(4) we would be interested in the subgroups of the unit quaternions group U. We also define the surjective 2:1 homomorphism $\Psi \colon U \to \mathrm{SO}(3)$ by $\Psi(u)(x) = ux\bar{u}$, whose kernel is $\{\pm 1\}$.

It is well-known [14] that the discrete subgroups of SO(3) are the cyclic groups, the dihedral groups and the pure symmetry groups of the platonic solids which are the tetrahedral, the octahedral and the icosahedral groups. The tetrahedral group $\mathcal{T}$ is the group of rotational symmetries of a tetrahedron, it has order 12 and it is isomorphic to A_4, the group of even permutations of 4 elements. The octahedral group $\mathcal{O}$ is the group of rotational symmetries of an octahedral (or a cube) and it is a group of order 24, isomorphic to S_4, the permutation group of 4 elements. The icosahedral group $\mathcal{I}$ is the group of rotational symmetries of a icosahedral (or a dodecahedral) and it has order 60, isomorphic to A_5.

Using the homomorphism Ψ, it is not hard to determine the discrete subgroups of the group of unit quaternions U. A complete description of how this is done can be found in [11].

Proposition 3.1. *Every finite subgroup of the unit quaternionic group U is conjugate to one of the following groups:*

$$\begin{aligned}
\mathbf{C_n} &= \left\{\cos\frac{2k\pi}{n} + \mathbf{k}\sin\frac{2k\pi}{n},\ k = 0,\dots,n-1\right\}\\
\mathbf{D_n} &= \mathbf{C_{2n}} \cup \mathbf{i}\mathbf{C_{2n}}\\
\mathbf{T} &= \bigcup_{k=0}^{2}\left(\frac{1}{2} + \frac{1}{2}\mathbf{i} + \frac{1}{2}\mathbf{j} + \frac{1}{2}\mathbf{k}\right)^k \mathbf{D_2}\\
\mathbf{O} &= \mathbf{T} \cup \frac{1}{\sqrt{2}}(\mathbf{1}+\mathbf{i})\mathbf{T}\\
\mathbf{I} &= \bigcup_{k=0}^{4}\left(\frac{1}{2\tau} + \frac{\tau}{2}\mathbf{i} + \frac{1}{2}\mathbf{j}\right)^k \mathbf{T}
\end{aligned}$$

where $\tau = \frac{\sqrt{5}+1}{2}$.

The binary polyhedral subgroups $\mathbf{T}, \mathbf{O}$ and $\mathbf{I}$ are called, respectively, the binary tetrahedral, octahedral and icosahedral subgroups of U and are twice the order of the corresponding polyhedral subgroup of SO(3).

Now, to find the discrete subgroups of SO(4) we use the $2:1$ homomorphism:

$$\begin{aligned}
&\Phi\colon U \times U \to \mathrm{SO}(4)\\
&\Phi(l,r)(x) = lx\bar{r}
\end{aligned}$$

with $\operatorname{Ker}\Phi = \{(1,1),(-1,-1)\}$. We denote by $[l,r] = \{(l,r),(-l,-r)\}$. Then we have an isomorphism $\varphi\colon U \times U/_{\mathbb{Z}_2} \cong \mathrm{SO}(4)$ with $\varphi([l,r]) = \{x \to lx\bar{r}\}$.

If $\sigma \subset \mathrm{SO}(4)$ is a discrete subgroup, then

$$L = \{l \in U | x \to lx\bar{r} \in \sigma\} \text{ and } R = \{r \in U | x \to lx\bar{r} \in \sigma\}$$

are subgroups of U. We notice that $\sigma \subseteq \Phi(L \times R)$ but the equality might not hold since it might be possible to find a pair $(l,r) \in L \times R$ such that $x \to lx\bar{r} \notin \sigma$. We define the subgroups

$$\begin{aligned}
L' &= \{l \in L | x \to lx \in \sigma\} = \{l \in L | (l,1) \in \sigma\}\\
R' &= \{r \in R | x \to x\bar{r} \in \sigma\} = \{r \in R | (1,r) \in \sigma\}
\end{aligned}$$

and there is an isomorphism between the quotient groups $L/_{L'}$ and $R/_{R'}$ given by $\psi(lL') = rR'$ such that $(l, r) \in \sigma$. The subgroup $\Phi(L' \times R')$ is normal in σ and $\sigma/_{\Phi(L'\times R')} \cong L/_{L'}$ and $R/_{R'}$. The subgroup σ depends on the isomorphism ψ between the quotient groups $L/_{L'}$ and $R/_{R'}$, different isomorphism possibly yielding non-conjugate subgroups in SO(4). We will denote the group σ by $(L/_{L'}; R/_{R'})_\psi$.

For example, $\mathbf{C_m}$ is a normal subgroup of $\mathbf{C_{mr}}$ and the quotient group $\mathbf{C_{mr}}/_{\mathbf{C_m}} \cong \mathbb{Z}_r$. The elements of $\mathbf{C_{mr}}/_{\mathbf{C_m}}$ are the cosets $\mathbf{p}^i\mathbf{C_m}$, $i = 0..r-1$, where $\mathbf{p} = \cos\frac{2\pi}{mr} + \mathbf{k}\sin\frac{2\pi}{mr}$ is a generator of $\mathbf{C_{mr}}$. If $\mathbf{q} = \cos\frac{2\pi}{nr} + \mathbf{k}\sin\frac{2\pi}{nr}$ is a generator of $\mathbf{C_{nr}}$, we have the isomorphism $\psi_s\colon \mathbf{C_{mr}}/_{\mathbf{C_m}} \to \mathbf{C_{nr}}/_{\mathbf{C_n}}$ defined by $\psi_s(\mathbf{p}^i\mathbf{C_m}) = \mathbf{q}^{si}\mathbf{C_n}, i = 0..r-1$. For each s such that $(s, r) = 1$ and $s < \frac{1}{2}r$, we get an isomorphism ψ_s that gives distinct subgroups $(\mathbf{C_{mr}}/_{\mathbf{C_m}}; \mathbf{C_{nr}}/_{\mathbf{C_n}})_{\psi_s}$. The subgroups of SO(4) of this form that do not contain the central rotation are seen to be $(\mathbf{C_{2mr}}/_{\mathbf{C_m}}; \mathbf{C_{2nr}}/_{\mathbf{C_n}})_{\psi_s}$, of order mnr with m and n odd. Also, extending the isomorphism between $\mathbf{C_{2mr}}/_{\mathbf{C_m}}$ and $\mathbf{C_{2nr}}/_{\mathbf{C_n}}$ to one between $\mathbf{D_{mr}}/\mathbf{C_m} = (\mathbf{C_{2mr}} \oplus \mathbf{iC_{2mr}})/\mathbf{C_m}$ and $\mathbf{D_{nr}}/\mathbf{C_n} = (\mathbf{C_{2nr}} \oplus \mathbf{iC_{2nr}})/\mathbf{C_n}$ by $\psi_s(\mathbf{ip}^j\mathbf{C_m}) = \mathbf{iq}^{sj}\mathbf{C_n}, j = 0...r-1$, we obtain the subgroup $(\mathbf{D_{mr}}/\mathbf{C_m}; \mathbf{D_{nr}}/\mathbf{C_n})_{\psi_s}$ of order $2mnr$, where m and n are odd.

Another subgroup of SO(4), of order 12, that does not contain the central rotation is

$$\mathbb{T} = (\mathbf{T}/\mathbf{C_1}; \mathbf{T}/\mathbf{C_1}) = (\mathbf{T}; \mathbf{T}) = \{[t, t] \mid t \in \mathbf{T}\}.$$

In the case case when $L = R = \mathbf{O}$ and $L' = R' = \mathbf{C_1}$, we obtain two non-conjugate groups, depending on the automorphism of $\mathbf{O}$ considered. If we take $\psi\colon \mathbf{O} \to \mathbf{O}$ to be the identical automorphism we obtain the subgroup $\mathbb{O} = (\mathbf{O}; \mathbf{O}) = \{[o, o], o \in \mathbf{O}\}$ of order 24. If we consider the automorphism $\psi\colon \mathbf{O} = \mathbf{T} \oplus (\mathbf{1} + \frac{1}{\sqrt{2}}\mathbf{i})\mathbf{T} \to \mathbf{O}$ with $\psi(o) = o$, if $o \in \mathbf{T}$ and $\psi(o) = -o$, if $o \in \frac{1}{\sqrt{2}}(\mathbf{1} + \mathbf{i})\mathbf{T}$, we obtain a different subgroup $\mathbb{O}^+ = \{[o, o], o \in \mathbf{T}$ and $[o, -o], o \in \frac{1}{\sqrt{2}}(\mathbf{1} + \mathbf{i})\mathbf{T}\}$, of order 24.

In the case when $L = R = \mathbf{I}$ and $L' = R' = \mathbf{C_1}$, we obtain again two non-conjugate subgroups of SO(4) which do not contain the central rotation. If we take $\psi\colon \mathbf{I} \to \mathbf{I}$ to be the identical automorphism, we obtain the subgroup $\mathbb{I} = (\mathbf{I}; \mathbf{I}) = \{[l, l], l \in \mathbf{I}\}$, of order 60. But we notice that all the elements of $\mathbf{I}$ are in the field of rational numbers over $\sqrt{5}$ and the change of sign of $\sqrt{5}$ interchanges $\pm\tau$ with $\mp\tau^{-1}$. If $\mathbf{p} \in \mathbf{I}$ is a quaternion we denote by $\mathbf{p}^+$ its image under this automorphism. Then $\mathbf{I}$ is interchanged with a group $\mathbf{I}^+ = \bigcup_{k=0}^{4} (\frac{\tau}{2} + \frac{1}{2\tau}\mathbf{i} + \frac{1}{2}\mathbf{j})^k\mathbf{T}$ and the two groups have in common $\mathbf{T}$. If we now consider the isomorphism $\psi\colon \mathbf{I}^+ \to \mathbf{I}, \psi(\mathbf{p}^+) = \mathbf{p}$ then we obtain a different subgroup $\mathbb{I}^+ = \{[\mathbf{r}^+, \mathbf{r}], \mathbf{r} \in \mathbf{I}\}$, of order 60. This group leaves no axis fixed and it can be shown to be the rotational symmetry group of a regular simplex in $\mathbb{E}_4$, with vertices at $\mathbf{1}$ and $\frac{1}{4}(-\mathbf{1} \pm \sqrt{5}\mathbf{i} \pm \sqrt{5}\mathbf{j} \pm \sqrt{5}\mathbf{k})$.

To conclude, we have the following:

Proposition 3.2. *The discrete subgroups of* $\mathrm{SO}(4)$ *that do not contain the central symmetry* $-I_4$ *are the following:*

1. $(\mathbf{C_{2mr}}/_{\mathbf{C_m}}; \mathbf{C_{2nr}}/_{\mathbf{C_n}})_{\psi_s}$, *of order* mnr, *where* m, n *odd*;
2. $(\mathbf{D_{mr}}/\mathbf{C_m}; \mathbf{D_{nr}}/\mathbf{C_n})_{\psi_s}$, *of order* $2mnr$, *where* m, n *odd*;
3. $\mathbb{T} = (\mathbf{T}/\mathbf{C_1}; \mathbf{T}/\mathbf{C_1}) = \{[t, t] \mid t \in \mathbf{T}\}$, *of order* 12;
4. $\mathbb{O} = (\mathbf{O}/\mathbf{C_1}; \mathbf{O}/\mathbf{C_1}) = \{[o, o] \mid o \in \mathbf{O}\}$, *of order* 24;
5. $\mathbb{O}^+ = (\mathbf{O}/\mathbf{C_1}; \mathbf{O}/\mathbf{C_1}) = \{[o, o] \mid o \in \mathbf{T}$ *and* $[o, -o], o \in \frac{1}{\sqrt{2}}(\mathbf{1} + \mathbf{i})\mathbf{T}\}$, *of order* 24;
6. $\mathbb{I} = (\mathbf{I}/\mathbf{C_1}; \mathbf{I}/\mathbf{C_1}) = \{[l, l] \mid l \in \mathbf{I}\}$, *of order* 60;
7. $\mathbb{I}^+ = (\mathbf{I}^+/\mathbf{C_1}; \mathbf{I}|\mathbf{C_1}) = \{[r^+, r] \mid r \in \mathbf{I}\}$, *of order* 60.

For complete proof of this proposition, the reader should consult [11] and [3].

3.2 Continuous stabilizer type

Any maximal torus in $\mathrm{SO}(4)$ is conjugate to the group:

$$\left\{ \begin{pmatrix} e^{i\theta_1} & 0 \\ 0 & e^{i\theta_2} \end{pmatrix},\ \theta_1, \theta_2 \in [0, 2\pi) \right\}.$$

We are looking to determine the orbits of the action (3.13) that have nontrivial stabilizer under the action of $\mathrm{SO}(4)$. In what follows, a positive dimensional stabilizer will be called a continuous stabilizer. The special Lagrangian 4-folds whose fundamental cubic at each point has stabilizer G, a subgroup of $\mathrm{SO}(4)$, will be said to have *stabilizer type* G. If the stabilizer G is a continuous or discrete subgroup of $\mathrm{SO}(4)$, we will say that the special Lagrangian 4-fold has *continuous* or *discrete stabilizer type*, respectively.

First, we are going to classify the harmonic cubic polynomials in 4 variables $\{x_1, x_2, x_3, x_4\}$ whose stabilizer is a continuous subgroup of $\mathrm{SO}(4)$.

Proposition 3.3. *The* SO(4)*-stabilizer of* $h \in \mathcal{H}_3(\mathbb{R}^4)$ *is a continuous subgroup of* SO(4) *if and only if* h *lies on the* SO(4)*-orbit of exactly one of the following polynomials:*

1. $0 \in \mathcal{H}_3(\mathbb{R}^4)$, *whose stabilizer is* SO(4);
2. $rx_1(x_1^2 - x_2^2 - x_3^2 - x_4^2)$ *for some* $r > 0$, *whose stabilizer is the subgroup* SO(3), *consisting of rotations in the* 3*-space* (x_2, x_3, x_4);
3. $r[(x_1^2 - x_2^2)x_3 + 2x_1x_2x_4]$, *for some* $r > 0$, *whose stabilizer is the subgroup* O(2) *generated by rotations by an arbitrary angle in the* (x_1, x_2)*-plane and twice that angle in the* (x_3, x_4)*-plane and the element* $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$;
4. $r(x_1^3 - 3x_1x_2^2)$ *for some* $r > 0$, *whose stabilizer is the subgroup* $\mathrm{SO}(2) \ltimes S_3$, *where* S_3 *is the symmetric group on* 3 *elements generated by the rotation by an angle of* $\frac{2\pi}{3}$ *in the* (x_1, x_2)*- plane and the element* $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$, *and* SO(2) *is the group of rotations in the* (x_3, x_4)*-plane*;
5. $r(x_1^3 - 3x_1x_2^2) + 3vx_1(x_1^2 + x_2^2 - 2x_3^2 - 2x_4^2)$ *for some* $v > 0$ *satisfying* $r \neq 3v$ *whose stabilizer is the* O(2)*-subgroup generated by rotations in the* (x_3, x_4)*-plane and the element* $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$;
6. $r(x_1^3 - 3x_1x_2^2) + s(3x_1^2x_2 - x_2^3) + 3vx_1(x_1^2 + x_2^2 - 2x_3^2 - 2x_4^2)$ *for some* $s > 0$ *and* $v > 0$ *whose stabilizer is the* SO(2)*-subgroup generated by rotations in the* (x_3, x_4)*-plane.*

Remark (special values)**.** The case $r = 3v$ of case 5 above reduces to case 2, when the stabilizer is SO(3).

Proof. Suppose $h \in \mathcal{H}_3(\mathbb{R}^4)$ has a nontrivial stabilizer $G \subseteq \mathrm{SO}(4)$. If $G = \mathrm{SO}(4)$, then $h = 0$ since $\mathcal{H}_3(\mathbb{R}^4)$ is an irreducible representation of SO(4). We suppose from now on that $h \neq 0$. Being a stabilizer, G is a closed subgroup of SO(4), therefore it is compact and has a finite number of components.

We suppose G is not discrete. Then, its identity component H is a closed connected subgroup of SO(4). The algebra $\mathfrak{h}$ of H is a subalgebra of $\mathfrak{so}(4)$. Using the 2:1 homomorphism $\Phi\colon U \times U \to \mathrm{SO}(4)$, $\Phi(u_1, u_2)(x) = u_1 x \bar{u}_2$ from Section 3.1, it is easy to see that $\mathfrak{so}(4) \cong$

$\mathfrak{so}(3)_+ \oplus \mathfrak{so}(3)_-$, where $\mathfrak{so}(3)_+$ and $\mathfrak{so}(3)_-$ are two different copies of $\mathfrak{so}(3)$ with intersection the 0 vector. Since $\dim \mathfrak{so}(4) = 6$, there are the following possibilities for the subalgebra $\mathfrak{h}$:

1) $\dim \mathfrak{h} = 5$. This is not possible for the following reason: $\mathfrak{h} \cap \mathfrak{so}(3)_+ \subset \mathfrak{so}(3)_+$ is a subalgebra of dimension at least 2 and $\mathfrak{so}(3)_+$ has no subalgebras of dimension 2. Therefore, $\mathfrak{h} \cap \mathfrak{so}(3)_+ = \mathfrak{so}(3)_+$ which implies that $\mathfrak{h} \supseteq \mathfrak{so}(3)_+$. Similarly, it can be shown that $\mathfrak{h} \supseteq \mathfrak{so}(3)_-$ and it follows from here that $\mathfrak{h} = \mathfrak{so}(4)$, which gives a contradiction.

2) $\dim \mathfrak{h} = 4$. Then $\mathfrak{h}_+ = \mathfrak{h} \cap \mathfrak{so}(3)_+$ is an ideal of dimension at least 1 in $\mathfrak{so}(3)_+$ and $\mathfrak{h}_- = \mathfrak{h} \cap \mathfrak{so}(3)_-$ is an ideal of dimension at least 1 in $\mathfrak{so}(3)_-$. Consider the projections $\pi_\pm \colon \mathfrak{h} \to \mathfrak{so}(3)_\pm$. Since Ker $\pi_\pm = \mathfrak{h}_\mp$, it follows that Ker $\pi_\pm$ can have dimension 1 or 3. If Ker π_- has dimension 1, π_- is onto and Ker π_+ has dimension 3. In this case, it follows that $\mathfrak{h}_+ \cong \mathfrak{so}(2)_+$ and we obtain that $\mathfrak{h} = \mathfrak{so}(2)_+ \oplus \mathfrak{so}(3)_-$. If the dimension of Ker π_- is 3, then the dimension of Ker π_+ is 1 and we obtain $\mathfrak{h} = \mathfrak{so}(3)_+ \oplus \mathfrak{so}(2)_-$. But $\mathfrak{so}(3)_\pm$ acts like the group SU(2) on the space of complexified harmonic cubics in four variables $\{z_1, z_2, \bar{z}_1, \bar{z}_2\}$, where $z_1 = x_1 + ix_2$, $z_2 = x_3 + ix_4$ and calculations show that this action does not preserve any nontrivial element. Therefore, in this case H does not preserve any nontrivial harmonic cubic.

3) $\dim \mathfrak{h} = 3$. In this case, one can show that, up to conjugacy, the only possibilities for $\mathfrak{h}$ are $\mathfrak{so}(3)_+$, $\mathfrak{so}(3)_-$ and $\operatorname{diag}(\mathfrak{so}(3)_+ \oplus \mathfrak{so}(3)_-)$. But, as discussed in 2) above, $\mathfrak{so}(3)_\pm$ does not preserve any cubic polynomial in 4 variables and consequently we can discard these cases.

We study now the case

$$\mathfrak{h} = \operatorname{diag}(\mathfrak{so}(3)_+ \oplus \mathfrak{so}(3)_-) = \{x_+ + x_-,\ x_+ \in \mathfrak{so}(3)_+,\ x_- \in \mathfrak{so}(3)_-\}.$$

We can see that, up to conjugacy,

$$\operatorname{diag}(\mathfrak{so}(3)_+ \oplus \mathfrak{so}(3)_-) = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -2c & 2b \\ 0 & 2c & 0 & -2a \\ 0 & -2b & 2a & 0 \end{pmatrix}, a, b, c \in \mathbb{R} \right\}.$$

Therefore, G can be either one of the groups:

$$G = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix}, A \in \mathrm{SO}(3) \right\} \quad \text{or} \quad G = \left\{ \begin{pmatrix} \det(A) & 0 \\ 0 & A \end{pmatrix},\ A \in \mathrm{O}(3) \right\}.$$

We can easily see that the cubic polynomials fixed by the identity component H are linear combinations of x_1^3 and $x_1(x_2^2+x_3^2+x_4^2)$. It is obvious that the only combination that would make the polynomial harmonic is $P = rx_1(x_1^2 - x_2^2 - x_3^2 - x_4^2)$, for some $r \neq 0$. One can verify that the full stabilizer of P is SO(3). By a rotation that reverses the x_1-axis, if necessary, we can assume that $r > 0$.

4) $\dim \mathfrak{h} = 2$. In this case, $\mathfrak{h} = \mathfrak{so}(2)_+ \oplus \mathfrak{so}(2)_-$ and H is conjugate to the maximal torus $H = \left\{ \begin{pmatrix} e^{i\theta_1} & 0 \\ 0 & e^{i\theta_2} \end{pmatrix},\ \theta_1, \theta_2 \in [0, 2\pi) \right\}$. It is easy to see that H does not stabilize any symmetric cubic in 4 variables.

5) $\dim \mathfrak{h} = 1$. In this case, one can show that the only 1-dimensional ideals in $\mathfrak{so}(4)$ are conjugate to:

$$\mathfrak{h}_{p,q} = \{(px, qx) |\ x \in \mathfrak{so}(2),\ p, q \in \mathbb{Z},\ (p, q) = 1\}.$$

It follows that the identity component $H_{p,q} = \left\{ \begin{pmatrix} R(p\theta) & 0 \\ 0 & R(q\theta) \end{pmatrix}, \theta \in \mathbb{R} \right\}$ consists of rotations of angle $p\theta$ in the (x_1, x_2)-plane and of angle $q\theta$ in the (x_3, x_4)-plane. We are looking for harmonic cubics in $\{x_1, x_2, x_3, x_4\}$ preserved by $H_{p,q}$, for some p and q integers.

Let V_n be the irreducible representation of SO(2) given by rotations of speed n: $e^{i\theta}.z = e^{ni\theta}z$, where $z \in \mathbb{C}$. In our case, the speed p representation V_p is given by the action on the (x_1, x_2)-plane: $e^{i\theta}.z_1 = e^{ip\theta}z_1$ and V_q is given by the action on the (x_3, x_4)-plane: $e^{i\theta}.z_2 = e^{iq\theta}z_2$, where $z_1 = x_1 + ix_2$ and $z_2 = x_3 + ix_4$.

Under the action of $H_{p,q}$, the space of symmetric polynomials in 4 variables $\mathcal{S}^3(\mathbb{R}^4)$ decomposes as:

$$\mathcal{S}^3(V_p \oplus V_q) = \mathcal{S}^3(V_p) \oplus (\mathcal{S}^2(V_p) \otimes \mathcal{S}^1(V_q)) \oplus (\mathcal{S}^1(V_p) \otimes \mathcal{S}^2(V_q)) \oplus \mathcal{S}^3(V_q).$$

But one can see that $\mathcal{S}^3(V_p) \cong V_{3p} \oplus V_p$. A basis in V_{3p} is $\{\operatorname{Re} z_1^3, \operatorname{Im} z_1^3\} = \{x_1^3 - 3x_1x_2^2, 3x_1^2x_2 - x_2^3\}$ and a basis in V_p is $\{\operatorname{Re}(z_1\bar{z}_1z_1), \operatorname{Im}(z_1\bar{z}_1z_1)\} = \{(x_1^2 + x_2^2)x_1, (x_1^2 + x_2^2)x_2\}$. Similarly, $\mathcal{S}^2(V_p) \cong V_{2p} \oplus V_0 \cong V_{2p} \oplus \mathbb{R}$ and we calculate that:

$$\begin{aligned}
\mathcal{S}^3(\mathbb{R}^4) &= (V_{3p} \oplus V_p) \oplus ((V_{2p} \oplus \mathbb{R}) \otimes V_q) \\
&\quad \oplus (V_p \otimes (V_{2q} \oplus \mathbb{R})) \oplus (V_{3q} \oplus V_q) \\
&= V_{3p} \oplus V_p \oplus V_{2p+q} \oplus V_{2p-q} \oplus V_{p+2q} \oplus V_{p-2q} \oplus V_{3q} \oplus V_q
\end{aligned}$$

where we used that $V_n \otimes V_m = V_{n+m} \oplus V_{n-m}$. This decomposition is irreducible and the action has a fixed vector if and only if one of the V's in the above direct sum is V_0, the 1-dimensional representation on which the action is trivial. This implies one of the following possibilities for the values of p and q:

$$p = 0,\ q = 0,\ p - 2q = 0,\ q - 2p = 0,\ p + 2q = 0 \text{ or } 2p + q = 0.$$

We note that p and q have symmetric roles since the planes (x_1, x_2) and (x_3, x_4) can be interchanged by an orthogonal transformation. Therefore, up to conjugation with an element in $\mathrm{SO}(4)$, the only possible cases are: $p = 0$, $q = 2p$ and $q = -2p$. In the case $q = 2p$, the fact that $(p, q) = 1$ implies that $p = 1$ and $q = 2$ and in the case $q = -2p$, we can take $p = 1, q = -2$. The conclusion is that, unless one of these conditions is satisfied, the group $H_{p,q}$ does not preserve any cubic symmetric polynomial in four variables. But since the stabilizer in $\mathrm{SO}(4)$ coincides with the stabilizer in $\mathrm{O}(4)$, then up to conjugacy in $\mathrm{O}(4)$, the last two cases are the same. We will study each of these cases separately.

a) $p = 1$ and $q = 2$. Then $H_{1,2} = \left\{ \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{2i\theta} \end{pmatrix}, \theta \in [0, 2\pi) \right\}$. If P is a complexified polynomial in the variables $\{z_1, z_2, \bar{z}_1, \bar{z}_2\}$, fixed by $H_{1,2}$, it is easy to see that P should lie in $V_2 \otimes V_1$, therefore $P = az_1^2\bar{z}_2 + b\bar{z}_1{}^2 z_2, a,\ b \in \mathbb{C}$. Now, P is a real harmonic polynomial if $b = \bar{a}$. So, any real harmonic cubic C preserved by this group is of the form:

$$C = \mathrm{Re}(az_1^2\bar{z}_2) = r[(x_1^2 - x_2^2)x_3 + 2x_1x_2x_4] + s[2x_1x_2x_3 - (x_1^2 - x_2^2)x_4],$$

with $r, s \in \mathbb{R}$. If $r^2 + s^2 \neq 0$, by applying a rotation of angle $\arctan(\frac{s}{r})$ in the (x_3, x_4)-plane, we can assume that $s = 0$. We can also assume that $r \geq 0$. The conclusion is that all the harmonic cubics in 4 variables stabilized by $H_{1,2}$ are on the $\mathrm{SO}(4)$-orbit of the cubic $h = r[(x_1^2 - x_2^2)x_3 + 2x_1x_2x_4]$. The full stabilizer of h can be shown to be the disconnected 2-piece subgroup $H_{1,2} \cup gH_{1,2} \subset \mathrm{SO}(4)$, where $g = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$. It is isomorphic to $\mathrm{O}(2)$, because $\mathrm{O}(2)$ is the only nonabelian 2-component compact group of dimension one.

b) If $p = 0$, we can consider $q = 1$. In this case, the identity component of G is $H_{0,1} = \left\{ \begin{pmatrix} I_2 & 0 \\ 0 & e^{i\theta} \end{pmatrix}, \theta \in \mathbb{R} \right\} \cong S^1$. A complexified cubic polynomial C, fixed by this group, should belong to $V_0 \otimes V_1$, therefore it should be a linear combination of $z_1z_2\bar{z}_2$, z_1^3 and $z_1^2\bar{z}_1$. Calculations show C is harmonic if and only if it is a linear combination of the harmonic polynomials $\{z_1^3, z_1^2\bar{z}_1 - 2z_1z_2\bar{z}_2\}$. Therefore, the fixed real harmonic cubic polynomials C in 4 variables are of the form

$$\begin{aligned} C &= r(x_1^3 - 3x_1x_2^2) + s(3x_1^2x_2 - x_2^3) \\ &\qquad + vx_1(x_1^2 + x_2^2 - 2x_3^2 - 2x_4^2) + ux_2(x_1^2 + x_2^2 - 2x_3^2 - 2x_4^2), \end{aligned}$$

with $r, s, u, v \in \mathbb{R}$. By making a rotation in the (x_1, x_2)-plane, we can suppose that $u = 0$.

It remains now to determine the full stabilizer of the polynomial

$$r(x_1^3 - 3x_1x_2^2) + s(3x_1^2x_2 - x_2^3) + vx_1(x_1^2 + x_2^2 - 2x_3^2 - 2x_4^2),$$

which we denote by G.

If $s \neq 0$ and $v \neq 0$, calculations show that the full stabilizer of h is just the identity component, so $G = \mathrm{SO}(2)$. By making some rotations, if necessary, we may assume that $s,\ v > 0$.

If $s = 0$, $v \neq 0$, and $r \neq 3v$, then $h = r(x_1^3 - 3x_1x_2^2) + vx_1(x_1^2 + x_2^2 - 2x_3^2 - 2x_4^2)$ and the full stabilizer is isomorphic to an $\mathrm{O}(2)$-subgroup, because we are also allowed to flip the signs of x_2 and x_4. In this case we can suppose that $v > 0$. In the case $s = 0$ and $r = 3v$, h becomes $6vx_1(x_1^2 - x_2^2 - x_3^2 - x_4^2)$ and we saw that this polynomial has stabilizer $\mathrm{SO}(3)$.

Finally, if $s = v = 0$ and $r > 0$, the polynomial $h = r(x_1^3 - 3x_1x_2^2)$ is preserved by the identity component S^1, but we can see that it is also fixed by the element of order 3, $A = \begin{pmatrix} e^{\frac{2\pi i}{3}} & 0 \\ 0 & I_2 \end{pmatrix}$ and by the element of order 2, $g = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ and both A and g do not belong to the identity component. The rotation A and the reflection g form a group isomorphic to S_3, the symmetric group in 3 elements. S_3 acts on the identity component S^1 by conjugation and it is easy to compute that the action of A on S^1 is trivial, while g acts by flipping the circle S^1, namely $g.\begin{pmatrix} I_2 & 0 \\ 0 & e^{i\theta} \end{pmatrix} = \begin{pmatrix} I_2 & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$. Using the exact sequence

$$0 \ \to \ S_1 \ \to \ G \ \to \ S_3 \ \to \ 0$$

one can verify that $G = S^1 \ltimes S_3$. This completes the proof of Proposition 3.3. q.e.d.

In the next step, we are going to analyze each of the cases given by Proposition 3.3 and classify the SL 4-folds in $\mathbb{C}^4$ whose fundamental cubic form has a continuous symmetry at each point.

3.2.1 SO(3)-symmetry

Example 1. In their paper [12], Harvey and Lawson found the following special Lagrangian submanifolds of $\mathbb{C}^4$, invariant under the diagonal action of SO(4) on $\mathbb{C}^4 = \mathbb{R}^4 \times \mathbb{R}^4$:

$$L_c = \{(s + it)\mathbf{u} \mid \mathbf{u} \in S^3 \subset \mathbb{R}^4, \mathrm{Im}(s + it)^4 = c\}, \tag{3.14}$$

where $c \in \mathbb{R}$ is a constant. The variety L_0 is an union of four special Lagrangian 4-planes and when $c \neq 0$, each component of L_c is diffeomorphic to $\mathbb{R} \times S^3$ and it is asymptotic to one pair of 4-planes in L_0.

Theorem 3.4. *If $L \subset \mathbb{C}^4$ is a connected nontrivial special Lagrangian submanifold whose cubic fundamental form has an* SO(3)*-symmetry at each point, then L is, up to rigid motion, an open subset of one of the Harvey-Lawson examples.*

Proof. In the above hypotheses, a trivial special Lagrangian submanifold is a special Lagrangian 4-plane. We can see that the fundamental cubic C of L lies on the orbit of the 0 cubic if and only L is trivial. Therefore, assume the fundamental cubic is not identically vanishing. The locus where C vanishes is a proper real-analytic subset of L, so its complement L^* is open and dense in L. By replacing L by its component L^*, we can assume without loss of generality that C is nowhere vanishing on L. Since the stabilizer of C_x is SO(3) for all $x \in L$, Proposition 3.3 implies the existence of a positive real-analytic function $r \colon L \to \mathbb{R}^+$ with the property that the equation

$$C = 3r\omega_1(\omega_1^2 - \omega_2^2 - \omega_3^2 - \omega_4^2)$$

defines an SO(3)-subbundle F of the bundle P_L of L-adapted coframes. On the subbundle F, the following identities hold:

$$\begin{pmatrix} \beta_{11} & \beta_{12} & \beta_{13} & \beta_{14} \\ \beta_{21} & \beta_{22} & \beta_{23} & \beta_{24} \\ \beta_{31} & \beta_{32} & \beta_{33} & \beta_{34} \\ \beta_{41} & \beta_{42} & \beta_{43} & \beta_{44} \end{pmatrix} = \begin{pmatrix} 3r\omega_1 & -r\omega_2 & -r\omega_3 & -r\omega_4 \\ -r\omega_2 & -r\omega_1 & 0 & 0 \\ -r\omega_3 & 0 & -r\omega_1 & 0 \\ -r\omega_4 & 0 & 0 & -r\omega_1 \end{pmatrix}. \tag{3.15}$$

Because F is an SO(3)-bundle, the forms α_{21}, a_{31} and α_{41} vanish mod $\{\omega_1, \omega_2, \omega_3, \omega_4\}$, meaning that there are functions t_{ij} on F such that:

$$\begin{aligned} \alpha_{21} &= t_{21}\omega_1 + t_{22}\omega_2 + t_{23}\omega_3 + t_{24}\omega_4 \\ \alpha_{31} &= t_{31}\omega_1 + t_{32}\omega_2 + t_{33}\omega_3 + t_{34}\omega_4 \\ \alpha_{41} &= t_{41}\omega_1 + t_{42}\omega_2 + t_{43}\omega_3 + t_{44}\omega_4. \end{aligned} \tag{3.16}$$

Also, there exist functions r_i, $i = 1, 2, 3, 4$ on F such that:

$$dr = \sum_{i=1}^{4} r_i \omega_i. \tag{3.17}$$

Substituting the relations (3.15), (3.16) and (3.17) into the identities

$$d\beta_{ij} = -\beta_{ik} \wedge \alpha_{kj} - \alpha_{ik} \wedge \beta_{kj} \tag{3.18}$$

and using the identities $d\omega_i = -\alpha_{ij} \wedge \omega_j$, one gets polynomial relations among r_i, t_{ij} which can be solved, leading to relations of the form:

$$\alpha_{21} = t\omega_2, \quad \alpha_{31} = t\omega_3, \quad \alpha_{41} = t\omega_4, \quad dr = -5rt\omega_1 \tag{3.19}$$

where we denoted $t_{22} = t_{33} = t_{44}$ by t.

Differentiating the last equation in (3.19), we get $0 = d(dr) = -5rd(t) \wedge \omega_1$, implying that there exits a function u on F such that

$$dt = u\omega_1. \tag{3.20}$$

Substituting the relations (3.19) and (3.20) into the identities

$$d\alpha_{ij} = -\alpha_{ik} \wedge \alpha_{kj} + \beta_{ik} \wedge \beta_{kj} \tag{3.21}$$

and expanding out using the identities $d\omega_i = -\alpha_{ij} \wedge \omega_j$ implies the relations:

$$\begin{aligned}
u &= 4r^2 - t^2 \\
d\alpha_{32} &= \alpha_{43} \wedge \alpha_{42} + (t^2 + r^2)\omega_3 \wedge \omega_2 \\
d\alpha_{42} &= a_{43} \wedge \alpha_{32} + (t^2 + r^2)\omega_4 \wedge \omega_2 \\
d\alpha_{43} &= \alpha_{42} \wedge \alpha_{32} + (t^2 + r^2)\omega_4 \wedge \omega_3.
\end{aligned}$$

Differentiating the last equations yields only identities.

So, $F \to L$ is a SO(3)-bundle on which the 1-forms $\{\omega_1, \omega_2, \omega_3, \omega_4, \alpha_{32}, \alpha_{42}, \alpha_{43}\}$ form a basis and they satisfy the structure equations:

$$
\begin{aligned}
d\omega_1 &= 0\\
d\omega_2 &= t\omega_1\wedge\omega_2+\alpha_{32}\wedge\omega_3+\alpha_{42}\wedge\omega_4\\
d\omega_3 &= t\omega_1\wedge\omega_3-\alpha_{32}\wedge\omega_2+\alpha_{43}\wedge\omega_4\\
d\omega_4 &= t\omega_1\wedge\omega_4-\alpha_{42}\wedge\omega_2-\alpha_{43}\wedge\omega_3\\
d\alpha_{32} &= \alpha_{43}\wedge\alpha_{42}+(t^2+r^2)\omega_3\wedge\omega_2\\
d\alpha_{42} &= -\alpha_{43}\wedge\alpha_{32}+(t^2+r^2)\omega_4\wedge\omega_2\\
d\alpha_{43} &= \alpha_{42}\wedge\alpha_{32}+(t^2+r^2)\omega_4\wedge\omega_3\\
dr &= -5rt\omega_1\\
dt &= (4r^2-t^2)\omega_1
\end{aligned}
\tag{3.22}
$$

and the exterior derivatives of these equations are identities.

The last two equations in (3.22) imply that

$$\frac{dr}{-5rt}=\frac{dt}{4r^2-t^2}=\omega_1.$$

This yields $d(r^{\frac{8}{5}}+t^2r^{-\frac{2}{5}})=0$ and since L and F are connected, there exists a function θ on L with $|\theta|<\frac{\pi}{8}$ such that:

$$
\begin{aligned}
r^{\frac{4}{5}} &= c^{\frac{4}{5}}\cos 4\theta\\
r^{-\frac{1}{5}}t &= c^{\frac{4}{5}}\sin 4\theta.
\end{aligned}
$$

From these last two equations and from last equation in (3.22), it follows that

$$\omega_1=\frac{d\theta}{c(\cos 4\theta)^{\frac{5}{4}}}.$$

Now, setting $\eta_i=c(\cos 4\theta)^{1/4}\omega_i$ for $i=2,3$ and 4 yields:

$$
\begin{aligned}
d\eta_2 &= -\alpha_{23}\wedge\eta_3-\alpha_{24}\wedge\eta_4\\
d\eta_3 &= -\alpha_{32}\wedge\eta_2-\alpha_{34}\wedge\eta_4\\
d\eta_4 &= -\alpha_{42}\wedge\eta_2-\alpha_{43}\wedge\eta_3\\
d\alpha_{32} &= -\alpha_{34}\wedge\alpha_{42}+\eta_3\wedge\eta_2\\
d\alpha_{42} &= -\alpha_{43}\wedge\alpha_{32}+\eta_4\wedge\eta_2\\
d\alpha_{43} &= -\alpha_{42}\wedge\alpha_{23}+\eta_4\wedge\eta_3.
\end{aligned}
$$

The above equations represent the structure equations of the metric of constant curvature 1 on the 3-sphere S^3.

Conversely, if $d\sigma^2$ is the metric of constant curvature 1 on S^3, then, on the product $L = (-\frac{\pi}{8}, \frac{\pi}{8}) \times S^3$, consider the quadratic form

$$g = \frac{d\theta^2 + \cos^2 4\theta d\sigma^2}{c^2(\cos 4\theta)^{5/2}}$$

and the cubic form

$$C = 3\frac{\cos^2 4\theta d\sigma^2 d\theta - d\theta^3}{c^2(\cos 4\theta)^{5/2}}.$$

The pair (g, C) satisfies the Gauss and Codazzi equations and by Theorem 2.4 this implies that (L, g) can be isometrically immersed as a special Lagrangian 4-fold in $\mathbb{C}^4$ inducing C as its fundamental cubic. For each value of c, there exists a unique corresponding special Lagrangian 4-fold. Since the structure equations (3.22) have an SO(4)-symmetry and Harvey and Lawson [12] found that all the special Lagrangian 4-folds in $\mathbb{C}^4$, invariant under the diagonal action of SO(4), can be written explicitly as (3.14), the conclusion of the theorem follows. q.e.d.

3.2.2 O(2)-symmetry

According to Proposition 3.3, there are two cases of O(2)-symmetry. The first one gives the following:

Theorem 3.5. *There is no connected nontrivial special Lagrangian 4-fold in $\mathbb{C}^4$ whose cubic fundamental form has an* O(2)*-symmetry at each point, where* O(2) *is the subgroup* $S^1 \cup g_0 S^1$ *with* $S^1 = \left\{ \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{2i\theta} \end{pmatrix}, \theta \in \mathbb{R} \right\}$ *and* $g_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$.

Proof. Let L be a special Lagrangian 4-fold that satisfies the hypotheses of the theorem and let C be its fundamental cubic. Proposition 3.3 implies that there exists a function $r\colon L \to \mathbb{R}_+$ for which the equation

$$C = 3r[(\omega_1^2 - \omega_2^2)\omega_3 + 2\omega_1\omega_2\omega_4]$$

defines an O(2)-subbundle $F \subset P_L$ of the L-adapted coframe bundle $P_L \to L$. On the subbundle F, the following identities hold:

$$\begin{pmatrix} \beta_{11} & \beta_{12} & \beta_{13} & \beta_{14} \\ \beta_{21} & \beta_{22} & \beta_{23} & \beta_{24} \\ \beta_{31} & \beta_{32} & \beta_{33} & \beta_{34} \\ \beta_{41} & \beta_{42} & \beta_{43} & \beta_{44} \end{pmatrix} = \begin{pmatrix} r\omega_3 & r\omega_4 & r\omega_1 & r\omega_2 \\ r\omega_4 & -r\omega_3 & -r\omega_2 & r\omega_1 \\ r\omega_1 & -r\omega_2 & 0 & 0 \\ r\omega_2 & r\omega_1 & 0 & 0 \end{pmatrix}. \tag{3.23}$$

Since F is an O(2)-bundle, the following relations hold: $\alpha_{31} = \alpha_{41} = \alpha_{32} = \alpha_{42} = \alpha_{43} - 2\alpha_{21} \equiv 0 \mod \{\omega_1, \omega_2, \omega_3, \omega_4\}$, meaning that there exist functions t_{ij} on F such that:

$$\begin{aligned} \alpha_{42} &= t_{11}\omega_1 + t_{12}\omega_2 + t_{13}\omega_3 + t_{14}\omega_4 \\ \alpha_{32} &= t_{21}\omega_1 + t_{22}\omega_2 + t_{23}\omega_3 + t_{24}\omega_4 \\ \alpha_{31} &= t_{31}\omega_1 + t_{32}\omega_2 + t_{33}\omega_3 + t_{34}\omega_4 \\ \alpha_{41} &= t_{41}\omega_1 + t_{42}\omega_2 + t_{43}\omega_3 + t_{44}\omega_4 \\ \alpha_{43} - 2\alpha_{21} &= t_{51}\omega_1 + t_{52}\omega_2 + t_{53}\omega_3 + t_{54}\omega_4. \end{aligned} \tag{3.24}$$

Moreover, there exist functions r_i on F, $i = 1, 2, 3$ and 4 such that

$$dr = \sum_{i=1}^{4} r_i \omega_i. \tag{3.25}$$

Substituting the relations (3.23), (3.24) and (3.25) into the identities

$$d\beta_{ij} = -\beta_{ik} \wedge \alpha_{kj} - \alpha_{ik} \wedge \beta_{kj} \tag{3.26}$$

and using the identities $d\omega_i = -\alpha_{ij} \wedge \omega_j$, one gets polynomial relations among r_i, t_{ij} which can be solved, leading to relations of the form:

$$\alpha_{31} = \alpha_{41} = \alpha_{32} = \alpha_{42} = \alpha_{43} - 2\alpha_{21} = 0, \quad dr = 0. \tag{3.27}$$

Substituting (3.23) and (3.27) into the identities $d\alpha_{ij} = -\alpha_{ik} \wedge \alpha_{kj} + \beta_{ik} \wedge \beta_{kj}$ yields $r = 0$, contrary to the hypothesis. q.e.d.

The second case of symmetry O(2) yields the following partial result:

Proposition 3.6. *There is a 2-parameter family of connected special Lagrangian 4-folds with the property that the symmetry group of the fundamental cubic at each point is isomorphic to* O(2)*, where* O(2) *is the subgroup* $S^1 \cup g_0 S^1$ *with* $S^1 = \left\{ \begin{pmatrix} I_2 & 0 \\ 0 & e^{i\theta} \end{pmatrix}, \theta \in \mathbb{R} \right\}$ *and* $g_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$.

Proof. Let L be a special Lagrangian 4-fold that satisfies the hypotheses of the theorem and let C be its fundamental cubic. Proposition 3.3 implies that there exist functions $r, v\colon L \to \mathbb{R}$, $v \geq 0, r \neq 3v$ and an O(2)-subbundle $F \subset P_L$ over L on which the following identity holds:

$$C = r(\omega_1^3 - 3\omega_1\omega_2^2) + 3v\omega_1(\omega_1^2 + \omega_2^2 - 2\omega_3^2 - 2\omega_4^2). \tag{3.28}$$

Since F is an O(2)-bundle, the following relations hold: $\alpha_{21} = \alpha_{31} = \alpha_{42} = \alpha_{32} = \alpha_{42} \equiv 0 \mod \{\omega_1, \omega_2, \omega_3, \omega_4\}$ and doing the differential analysis as in the previous cases we obtain the following structure equations on the subbundle F:

$$\begin{aligned} d\omega_1 &= (r-3v)t_2\omega_1 \wedge \omega_2 \\ d\omega_2 &= (r-v)t_1\omega_1 \wedge \omega_2 \\ d\omega_3 &= 2vt_1\omega_1 \wedge \omega_3 + 2vt_2\omega_2 \wedge \omega_3 + \alpha_{43} \wedge \omega_4 \\ d\omega_4 &= 2vt_1\omega_1 \wedge \omega_4 + 2vt_2\omega_2 \wedge \omega_4 - \alpha_{43} \wedge \omega_3 \\ d(\alpha_{43}) &= 4v^2(t_1^2 + t_2^2 + 1)\omega_4 \wedge \omega_3 \\ dr &= -t_1(3r^2 - rv + 6v^2)\omega_1 + t_2(r-3v)(3r-2v)\omega_2 \\ dv &= -t_1 v(7v + r)\omega_1 + t_2 v(r-3v)\omega_2 \\ d(t_1) &= [r(t_1^2 + t_2^2 + 1) + v(5t_1^2 - 3t_2^2 + 5)]\omega_1 \\ d(t_2) &= 8vt_1t_2\omega_1 + (v-r)(t_1^2 + t_2^2 + 1)\omega_2 \end{aligned} \tag{3.29}$$

for some functions t_1, t_2. Differentiating these equations yields only identities.

We were not able to integrate completely the structure equations and find the family of special Lagrangian 4-folds that are solutions to these equations. One thing we could observe is that the generic solution has rank 2, since the following relations hold between the parameters r, v, t_1 and t_2:

$$d\left(\frac{(t_1^2 + t_2^2 + 1)v^{\frac{4}{5}}(r-3v)}{(r-v)^{\frac{3}{5}}}\right) = 0 \qquad \text{and}$$
$$d\left(\frac{(t_2^2(r-3v) + (r-v)(t_2^2+1))v^{\frac{7}{5}}}{(r-v)^{\frac{4}{5}}}\right) = 0.$$

The symmetry group of the solutions is three dimensional and it is either SO(3) or SO(2) $\times\, \mathbb{R}^2$, depending on the values of the parameters r, v, t_1 and t_2.

Remark. In principle, the structure equations can be integrated using the reduction process for special Lagrangian submanifolds with symmetries, by solving the ODE associated to it, as in [7]. The solution would be in terms of the Jacobi elliptic functions. As for now, we do not have an explicit integral yet.

3.2.3 $\mathbf{SO(2)} \ltimes S_3$-symmetry

Theorem 3.7. *Suppose that $L \subset \mathbb{C}^4$ is a connected special Lagrangian 4-fold with the property that its fundamental cubic at each point has an $\mathrm{SO}(2) \ltimes S_3$-symmetry. Then L is congruent to a product $\Sigma \times \mathbb{R}^2$, where $\Sigma \subset \mathbb{C}^2$ is a holomorphic curve.*

Proof. Let L be a special Lagrangian 4-fold that satisfies the hypotheses of the theorem and let C be its fundamental cubic. Proposition 3.3 implies that there exists a function $r\colon L \to \mathbb{R}_+$ for which the equation

$$C = r(\omega_1^3 - 3\omega_1\omega_2^2)$$

defines an $\mathrm{SO}(2) \ltimes S_3$-subbundle $F \subset P_L$ of the L-adapted coframe bundle $P_L \to L$, subbundle on which the 1-forms $\omega_1, \omega_2, \omega_3, \omega_4$ and α_{43} form a basis. Similar calculations as in previous cases show that the structure equations on F are:

$$\begin{aligned}
(3.30) \qquad & d\omega_1 = t_1\omega_1 \wedge \omega_2, \quad d\omega_2 = t_2\omega_1 \wedge \omega_2, \quad d\omega_3 = \alpha_{43} \wedge \omega_4, \\
& d\omega_4 = -\alpha_{43} \wedge \omega_3, \quad d\alpha_{43} = 0 \\
& dr = -3rt_2\omega_1 + 3rt_1\omega_2 \\
& dt_1 = -u_2\omega_1 + (t_1^2 + t_2^2 - 2r^2 + u_1)\omega_2 \\
& dt_2 = u_1\omega_1 + u_2\omega_2
\end{aligned}$$

for some functions u_1, u_2. Differentiation of these equations does not lead to new relations among the quantities because the system becomes involutive, according to Cartan-Kähler Theorem. This is seen by computing the Cartan characters: $s_1 = 2$, $s_2 = s_3 = s_4 = 0$ and noticing that the space of integral elements at each point is parametrized by 2 parameters u_1, u_2.

We are looking to integrate the above equations and find the family of special Lagrangian 4-folds that satisfy the hypothesis of the theorem. The Cartan-Kähler analysis tells us that the solution should depend on 2 functions of one variable.

From the above structure equations, we can see that $\omega_1 = \omega_2 = 0$ and $\omega_3 = \omega_4 = 0$ define integrable 2-plane fields on L. The 2-dimensional leaves of the 2-plane field Γ_1 defined by $\omega_3 = \omega_4 = 0$ are congruent along Γ_2, the codimension 2 foliation defined by $\omega_1 = \omega_2 = 0$. This is clear since $dt_1 = dt_2 \equiv 0 \mod \{\omega_1, \omega_2\}$ and therefore the structure equations of Γ_1 are:

$$d\omega_1 = t_1\omega_1 \wedge \omega_2, \ d\omega_2 = t_2\omega_1 \wedge \omega_2$$

where t_1, t_2 are constant along Γ_2. Also, the third to fifth equations in (3.30) imply that the leaves of the foliation Γ_2 are 2-planes which are congruent along Γ_1 since $d(e_3 \wedge e_4) = 0$ and the 2-planes are real, spanned by $\{e_3, e_4\}$. Therefore, L is a product $L = \Sigma \times \mathbb{R}^2$ where $\Sigma \subset \mathbb{C}^2$ is a surface. In order for L to be a special Lagrangian 4-fold, Σ should be a holomorphic curve with respect to a certain unique complex structure on $\mathbb{C}^2$. This is because of the following argument: Choose coordinates $z_k = x_k + iy_k, \ k = 1...4$ on $L = \Sigma \times \mathbb{R}^2$. Then L is special Lagrangian if and only if the 2-forms $dx_1 \wedge dy_1 + dx_2 \wedge dy_2$ and $dx_1 \wedge dy_2 + dy_1 \wedge dx_2$ each vanish when pulled back to Σ. But

$$\begin{aligned}&(dx_1 \wedge dy_1 + dx_2 \wedge dy_2) + i(dx_1 \wedge dy_2 + dy_1 \wedge dx_2)\\ &= (dx_1 - idx_2) \wedge (dy_1 + idy_2) = du \wedge dv\end{aligned}$$

where $u = x_1 - ix_2$ and $v = y_1 + iy_2$ are a different set of complex coordinates on $\mathbb{C}^2$. Then $\Sigma \subset \mathbb{C}^2$ is special Lagrangian if and only if $du \wedge dv \mid_\Sigma = 0$, which says that Σ is a holomorphic curve in $\mathbb{C}^2$ with respect to the complex coordinates (u, v) on $\mathbb{C}^2$. q.e.d.

3.2.4 SO(2)-symmetry

Theorem 3.8. *Suppose that $L \subset \mathbb{C}^4$ is a connected special Lagrangian 4-fold with the property that its fundamental cubic has an* SO(2)*-symmetry at each point. Then L is invariant under an* SO(3)*-action, whose orbits are 2-spheres, and the surface we obtain in the quotient M of $\mathbb{C}^4$ by this action is a pseudo-holomorphic curve with respect to a natural almost complex structure on this quotient M. Conversely, the pre-image of any pseudo-holomorphic curve in M gives a special Lagrangian 4-fold whose fundamental cubic has an* SO(2)*-symmetry at each point.*

Proof. Let L be a special Lagrangian 4-fold that satisfies the hypotheses of the theorem and let C be its fundamental cubic. Proposition 3.3 implies that there exist functions $r\colon L \to \mathbb{R}$, $s\colon L \to R_+$ and $v\colon L \to \mathbb{R}_+$ for which the equation

$$C = r(\omega_1^3 - 3\omega_1\omega_2^2) + s(3\omega_1^2\omega_2 - \omega_2^3) + 3v\omega_1(\omega_1^2 + \omega_2^2 - 2\omega_3^2 - 2\omega_4^2)$$

defines an SO(2)-subbundle $F \subset P_L$ of the L-adapted coframe bundle $P_L \to L$. The 1-forms $\omega_1, \omega_2, \omega_3, \omega_4$ and α_{43} form a basis and satisfy the structure equations:

(3.31)
$$\begin{aligned}
d\omega_1 &= [-t_1 s + (r-3v)t_2]\omega_1 \wedge \omega_2, \quad d\omega_2 = [t_2 s + (r-v)t_1]\omega_1 \wedge \omega_2\\
d\omega_3 &= 2t_1 v\omega_1 \wedge \omega_3 + 2t_2 v\omega_2 \wedge \omega_3 + \alpha_{43} \wedge \omega_4,\\
d\omega_4 &= 2t_1 v\omega_1 \wedge \omega_4 + 2t_2 v\omega_2 \wedge \omega_4 - \alpha_{43} \wedge \omega_3\\
d\alpha_{43} &= -4v^2(t_1^2 + t_2^2 + 1)\omega_3 \wedge \omega_4\\
dr &= [t_1(-6v^2 + vr - 3s^2 - 3r^2) - 11t_2 vs - t_4]\omega_1\\
&\quad + [-t_1 vs + t_2(6v^2 - 11vr + 3r^2 + 3s^2) + t_3]\omega_2\\
ds &= t_3\omega_1 + t_4\omega_2\\
dv &= -v[t_1(r+7v) + t_2 s]\omega_1 + v[-t_1 s + (r-3v)t_2]\omega_2\\
dt_1 &= [t_1^2(r+5v) + t_2^2(r-3v) + r + 5v]\omega_1 + [s(t_1^2 + t_2^2) + s]\omega_2\\
dt_2 &= [8vt_1t_2 + s + s(t_1^2 + t_2^2)]\omega_1 + (v-r)[(t_1^2 + t_2^2) + 1]\omega_2\\
dt_3 &= u_1\omega_1 + [u_2 + t_2t_3(r-3v) + t_1t_4(r-v) - t_1t_3 s + t_2t_4 s]\omega_2\\
dt_4 &= u_2\omega_1 + (-u_1 - 9t_1^2 s^3 - 6s^3 + 24rsv - 6r^2 s - 18v^2 s\\
&\quad - 60t_1t_2 vs^2 + 90rsvt_2^2 + 3vt_1t_3 + 30t_1^2 vrs\\
&\quad - 9t_2^2 s^3 - 7rt_1t_3 - 9t_2^2 r^2 s - 21t_1^2 v^2 s - 7t_2t_3 s + 7t_2t_4 r\\
&\quad - 25t_2t_4 v - 141t_2^2 v^2 s - 9t_1^2 r^2 s - 7t_1t_4 s)\omega_2
\end{aligned}$$

for some functions u_1, u_2. The above system is in involution, so differentiation of these equations does not lead to new relationships among quantities.

From the above structure equations, we can see that $\omega_1 = \omega_2 = 0$ and $\omega_3 = \omega_4 = 0$ define integrable 2-plane fields on L. The 2-dimensional leaves of the 2-plane field Γ_1 defined by $\omega_1 = \omega_2 = 0$ are 2-spheres. This is clear since the structure equations of the leaves of Γ_1 are the structure equations of a 2-dimensional sphere of constant radius $4v^2(t_1^2 + t_2^2 + 1)$:

$$d\omega_3 = -\alpha_{34} \wedge \omega_4,\ d\omega_4 = \alpha_{34} \wedge \omega_3,\ d\alpha_{34} = 4v^2(t_1^2 + t_2^2 + 1)\omega_3 \wedge \omega_4$$

and t_1, t_2, v are constant along Γ_1 since $dt_1 \equiv dt_2 \equiv dv \equiv 0 \mod \{\omega_1, \omega_2\}$. Therefore L is foliated by non-congruent spheres.

The 2-dimensional leaves of the other foliation Γ_2, defined by $\omega_3 = \omega_4 = 0$, are congruent. This follows from the structure equations

$$d\omega_1 = [-t_1 s + (r - 3v)t_2]\omega_1 \wedge \omega_2, \quad d\omega_2 = [t_2 s + (r - v)t_1]\omega_1 \wedge \omega_2$$

and the fact that $dr \equiv dv \equiv dt_1 \equiv dt_2 \equiv 0 \mod \{\omega_1, \omega_2\}$.

Also, the structure equations imply $d(e_1 \wedge e_2 \wedge Je_1 \wedge Je_2) = 0 \mod \{\omega_3, \omega_4\}$. Therefore the complex 2-plane (e_1, e_2, Je_1, Je_2) is constant along each leaf of the Γ_2-foliation and each such leaf lies in an affine plane parallel to this 2-plane.

If we let $\omega_{21} = [-t_1 s + (r-3v)t_2]\omega_1 + [t_2 s + (r-v)t_1]\omega_2$, the structure equation for the Γ_2 leaves can be written as:

$$d\omega_1 = \omega_{21} \wedge \omega_2, \ d\omega_2 = -\omega_{21} \wedge \omega_1, \ d\omega_{21} = 2(r^2 + s^2 - v^2)\omega_1 \wedge \omega_2.$$

This shows that the leaves of the Γ_2 foliation are congruent surfaces of Gauss curvature $2(v^2 - r^2 - s^2)$, lying in the affine complex 2-plane (e_1, e_2, Je_1, Je_2).

Computations show that the structure equations are invariant under an SO(3)-rotation about some point in $\mathbb{C}^4$. Therefore, the solutions should be special Lagrangian 4-folds that are invariant under the subgroup SO(3), as it sits naturally in SO(4) and hence in SU(4). The orbits of the SO(3)-action are 2-spheres.

We look now for special Lagrangian 4-folds L, invariant under the action of SO(3). Let

$$\mathbf{z} = \begin{pmatrix} \mathbf{x} + i\mathbf{y} \\ x_4 + iy_4 \end{pmatrix}, \ \mathbf{x} = (x_1, x_2, x_3), \mathbf{y} = (y_1, y_2, y_3) \in \mathbb{R}^3$$

denote the coordinates on $\mathbb{C}^4$. The subgroup SO(3) acts diagonally by rotation in $\mathbf{x}$ and $\mathbf{y}$,

$$A \cdot \begin{pmatrix} \mathbf{x} + i\mathbf{y} \\ x_4 + iy_4 \end{pmatrix} = \begin{pmatrix} A\mathbf{x} + iA\mathbf{y} \\ x_4 + iy_4 \end{pmatrix}, \ A \in \mathrm{SO}(3).$$

Let X_1, X_2, X_3 be the infinitesimal generators of SO(3), where

$$\begin{aligned} X_1 &= x_2 \frac{\partial}{\partial x_3} - x_3 \frac{\partial}{\partial x_2} + y_2 \frac{\partial}{\partial y_3} - y_3 \frac{\partial}{\partial y_2} \\ X_2 &= x_3 \frac{\partial}{\partial x_1} - x_1 \frac{\partial}{\partial x_3} + y_3 \frac{\partial}{\partial y_1} - y_1 \frac{\partial}{\partial y_3} \\ X_3 &= x_1 \frac{\partial}{\partial x_2} - x_2 \frac{\partial}{\partial x_1} + y_1 \frac{\partial}{\partial y_2} - y_2 \frac{\partial}{\partial y_1}. \end{aligned}$$

The 4-fold L is invariant under the flow of X_i, $i = 1, 2, 3$, so $(X_i \lrcorner \omega) \mid_L = 0, i = 1, 2, 3$, where $\omega = d\mathbf{x} \cdot d\mathbf{y} + dx_4 \wedge dy_4$ is the symplectic form and

$$d\mathbf{x} \cdot d\mathbf{y} := dx_1 \wedge dy_1 + dx_2 \wedge dy_2 + dx_3 \wedge dy_3.$$

It is easy to calculate that

$$(X_1 \lrcorner \omega) \mid_L = d(x_2y_3 - x_3y_2)$$

which implies that $x_2y_3 - x_3y_2 = c_1$, where $c_1 \in \mathbb{R}$ is a constant. Similarly, we can show that

$$x_3y_1 - x_1y_3 = c_2, \; x_1y_2 - x_2y_1 = c_3.$$

From here it follows that

$$\mathbf{x} \times \mathbf{y} = \mathbf{c} = (c_1, c_2, c_3),$$

where $\mathbf{c} \in \mathbb{R}^3$ is a constant vector.

If $c \neq 0$, then $\mathbf{x}, \mathbf{y}$ are linearly independent and therefore the stabilizer of a point on the orbit is trivial. This implies that the orbit has dimension 3, but we know that the orbits are 2 dimensional spheres. It follows that $c = 0$, i.e., $\mathbf{x}$ and $\mathbf{y}$ are linearly dependent. So, L lies in the 6-manifold $\Sigma \subset \mathbb{C}^4$ on which the coordinates are given by

$$\mathbf{z} = \begin{pmatrix} (x + iy)\mathbf{u} \\ x_4 + iy_4 \end{pmatrix}, \; x, y \in \mathbb{R}, \; \mathbf{u} = (u_1, u_2, u_3) \in S^2.$$

It is easy to compute that

$$\begin{aligned} \omega \mid_\Sigma &= d(xu_1) \wedge d(yu_1) + d(xu_2) \wedge d(yu_2) + d(xu_3) \wedge d(yu_3) + dx_4 \wedge dy_4 \\ &= dx \wedge dy + dx_4 \wedge dy_4. \end{aligned}$$

Dividing out by the SO(3)-action on Σ, we obtain in the quotient a 4-dimensional manifold X^4, with coordinates (x, y, x_4, y_4). The leaves of the $\omega_3 = \omega_4 = 0$ foliation are 2-dimensional surfaces M^2. We calculate now the pullback of the volume form to Σ. Denote $z = x + iy$ and $w = x_4 + iy_4$. So,

$$\begin{aligned} \Omega \mid_\Sigma &= dz_1 \wedge dz_2 \wedge dz_3 \wedge dz_4 \mid_\Sigma \\ &= d(zu_1) \wedge d(zu_2) \wedge d(zu_3) \wedge dw \\ &= z^2(u_3du_1 \wedge du_2 + u_1du_2 \wedge du_3 + u_2du_3 \wedge du_1) \wedge dz \wedge dw \\ &= \frac{1}{3}d(z^3) \wedge dw \wedge d\sigma \end{aligned}$$

where $d\sigma = u_3 du_1 \wedge du_2 + u_1 du_2 \wedge du_3 + u_2 du_3 \wedge du_1$ is the area form of the 2-sphere S^2.

Then $L \subset \Sigma$ is special Lagrangian if and only if the 2-forms

$$\alpha = dx \wedge dy + dx_4 \wedge dy_4 = \frac{i}{2}(dz \wedge d\bar{z} + dw \wedge d\overline{w})$$
$$\beta = \mathrm{Im}(\frac{1}{3}d(z^3) \wedge dw) = -\frac{i}{6}(d(z^3) \wedge dw - d(\bar{z}^3) \wedge d\overline{w})$$

each vanish when pulled back to $M^2 \subset X^4$. But:

$$\alpha \wedge \alpha = -\frac{1}{2}(dz \wedge d\bar{z} \wedge dw \wedge d\overline{w})$$
$$\beta \wedge \beta = \frac{1}{18}(d(z^3) \wedge dw \wedge d(\bar{z}^3) \wedge d\overline{w}) = \frac{1}{2}((z\bar{z})^2 dz \wedge dw \wedge d\bar{z} \wedge d\overline{w}).$$

Rescaling β by dividing it by $z\bar{z}$

$$\widetilde{\beta} = -\frac{i}{6z\bar{z}}(d(z^3) \wedge dw - d(\bar{z}^3) \wedge d\overline{w})$$

we get that $(\alpha + i\widetilde{\beta})^2 = 0$, so this form is decomposable and it is easy to compute that

$$\alpha + i\widetilde{\beta} = \frac{i}{2z\bar{z}}(\xi_1 \wedge \xi_2),$$

where $\xi_1 = zdz - i\bar{z}d\overline{w}$ and $\xi_2 = zd\bar{z} - izdw$. The forms ξ_1, ξ_2 form a system which is not integrable since

$$d\xi_1 = \frac{z}{\bar{z}} dw \wedge d\overline{w} \neq 0 \mod \{\xi_1, \xi_2\}.$$

In fact there is no combination of the forms ξ_1, ξ_2 that is integrable.

Since $\alpha + i\widetilde{\beta} \mid_M = 0$, it implies that M^2 is a pseudo-holomorphic curve in X^4, with respect to a certain almost complex structure, which is not integrable. Conversely, every almost complex surface in X^4 lifts to a special Lagrangian 4-fold $L \subset \Sigma^6 \subset \mathbb{C}^4$. q.e.d.

3.3 Discrete stabilizer type

Next, we are analyzing the case of discrete stabilizer type. Suppose that the stabilizer G of the fundamental cubic of a special Lagrangian 4-fold is a finite subgroup of SO(4). If g is an element of G, then g is conjugate to an element in the maximal torus of SO(4):

$$\left\{ \begin{pmatrix} e^{2\pi i r} & 0 \\ 0 & e^{2\pi i s} \end{pmatrix}, r \in \mathbb{Q},\ s \in \mathbb{Q},\ r, s < 1 \right\}.$$

The following result tells us when there exists a harmonic cubic in 4 variables fixed by a nontrivial element g in the maximal torus:

Proposition 3.9. *The element* $g = \begin{pmatrix} e^{2\pi i r} & 0 \\ 0 & e^{2\pi i s} \end{pmatrix}$, *where* $r \in \mathbb{Q}, s \in \mathbb{Q},\ r, s < 1$, *fixes a nontrivial harmonic cubic in four variables* (x_1, x_2, x_3, x_4) *if and only if at least one of the following conditions is satisfied:*

1. $3r \in \mathbb{Z}$, *when the fixed harmonic cubics contain linear combinations of* $\{x_1^3 - 3x_1x_2^2,\ 3x_1^2x_2 - x_2^3\}$;
2. $r \in \mathbb{Z}$, *when the fixed harmonic cubics contain linear combinations of* $\{x_1^3 - 3x_1x_2^2,\ 3x_1^2x_2 - x_2^3,\ x_1(x_1^2 + x_2^2 - 2x_3^2 - 2x_4^2),\ x_2(x_1^2 + x_2^2 - 2x_3^2 - 2x_4^2)\}$;
3. $2r + s \in \mathbb{Z}$, *when the fixed harmonic cubics contain linear combinations of* $\{(x_1^2 - x_2^2)x_3 - 2x_1x_2x_4,\ (x_1^2 - x_2^2)x_4 + 2x_1x_2x_3\}$;
4. $2r - s \in \mathbb{Z}$, *when the fixed harmonic cubics contain linear combinations of* $\{(x_1^2 - x_2^2)x_3 + 2x_1x_2x_4,\ (x_1^2 - x_2^2)x_4 - 2x_1x_2x_3\}$;
5. $2s + r \in \mathbb{Z}$, *when the fixed harmonic cubics contain linear combinations of* $\{(x_3^2 - x_4^2)x_1 - 2x_2x_3x_4,\ (x_3^2 - x_4^2)x_2 + 2x_1x_3x_4\}$;
6. $2s - r \in \mathbb{Z}$, *when the fixed harmonic cubics contain linear combinations of* $\{(x_3^2 - x_4^2)x_1 + 2x_2x_3x_4,\ (x_3^2 - x_4^2)x_2 - 2x_1x_3x_4\}$;
7. $3s \in \mathbb{Z}$, *when the fixed harmonic cubics contain linear combinations of* $\{x_3^3 - 3x_3x_4^2,\ 3x_3^2x_4 - x_4^3\}$;
8. $s \in \mathbb{Z}$, *when the fixed harmonic cubics contain linear combinations of* $\{x_3^3 - 3x_3x_4^2,\ 3x_3^2x_4 - x_4^3,\ x_3(2x_1^2 + 2x_2^2 - x_3^2 - x_4^2),\ x_4(2x_1^2 + 2x_2^2 - x_3^2 - x_4^2)\}$.

Proof. Let $\mathcal{P}^3_{\mathbb{C}} = \mathcal{P}^3(z_1, z_2, \overline{z_1}, \overline{z_2})$ be the space of complexified cubic polynomials, in the variables $(z_1, z_2, \overline{z_1}, \overline{z_2})$, where $z_1 = x_1 + ix_2,\ z_2 = x_3 + ix_4$. The maximal torus in SO(4) acts on $\mathcal{H}^3_{\mathbb{C}} = \mathcal{H}^3(z_1, z_2, \overline{z_1}, \overline{z_2})$, the space of complexified harmonic cubics in 4 variables $(z_1, z_2, \overline{z_1}, \overline{z_2})$, as follows:

$$\left(\begin{smallmatrix} e^{2\pi i r} & 0 \\ 0 & e^{2\pi i s} \end{smallmatrix}\right).P(z_1, z_2, \overline{z_1}, \overline{z_2}) = P(z_1^*, z_2^*, \overline{z_1^*}, \overline{z_2^*}),\ P \in \mathcal{H}^3_{\mathbb{C}}$$

where $z_1^* = e^{2\pi i r} z_1$, $z_2^* = e^{2\pi i s} z_2$. Under this action, the space $\mathcal{H}^3_{\mathbb{C}}$ decomposes as follows:

$$\mathcal{H}^3_{\mathbb{C}} = \mathcal{H}^3(z_1, z_2) \oplus \mathcal{H}(\mathcal{P}^2(z_1, z_2) \otimes \mathcal{P}^1(\overline{z_1}, \overline{z_2})) \oplus \mathcal{H}(\mathcal{P}^1(z_1, z_2) \otimes \mathcal{P}^2(\overline{z_1}, \overline{z_2})) \oplus \mathcal{H}^3(\overline{z_1}, \overline{z_2}).$$

A basis for the space $\mathcal{H}^3(z_1, z_2)$ is given by the polynomials $\{z_1^3, z_1^2 z_2, z_1 z_2^2, z_2^3\}$. Since $g.z_1 = e^{2\pi i r} z_1$ and $g.z_2 = e^{2\pi i s} z_2$, it follows that $g.z_1^3 = e^{6\pi i r} z_1^3$, $g.z_1^2 z_2 = e^{2\pi i(2r+s)} z_1^2 z_2$, $g.z_1 z_2^2 = e^{2\pi(r+2s)} z_1 z_2^2$ and $g.z_2^3 = e^{6\pi i s} z_2^3$. This further implies that unless $e^{6\pi i r} = 1$, $e^{2\pi i(2r+s)} = 1$, $e^{2\pi i(r+2s)} = 1$ or $e^{6\pi i s} = 1$, there is no fixed element in the space $\mathcal{H}^3(z_1, z_2)$. The above conditions are equivalent to $3r \in \mathbb{Z}$, $2r + s \in \mathbb{Z}$, $r + 2s \in \mathbb{Z}$ or $3s \in \mathbb{Z}$. Therefore, $\mathcal{H}^3(z_1, z_2)$ decomposes into the following four weight spaces:

$$\mathcal{H}^3(z_1, z_2) = V_{(3,0)} \oplus V_{(2,1)} \oplus V_{(1,2)} \oplus V_{(0,3)}.$$

All these weight spaces have multiplicity 1 and a basis in $V_{(3,0)}, V_{(2,1)}, V_{(1,2)}, V_{(0,3)}$ is given by the harmonic polynomials $z_1^3, z_1^2 z_2, z_1 z_2^2$ and z_2^3 respectively.

We analyze now the fixed elements for the space $\mathcal{H}(\mathcal{P}^2(z_1, z_2) \otimes \mathcal{P}^1(\overline{z_1}, \overline{z_2}))$ of harmonic polynomials in $\mathcal{P}^2(z_1, z_2) \otimes \mathcal{P}^1(\overline{z_1}, \overline{z_2})$. It is easy to see that a basis in the space $\mathcal{H}(\mathcal{P}^2(z_1, z_2) \otimes \mathcal{P}^1(\overline{z_1}, \overline{z_2}))$ is given by the harmonic cubics

$$\{z_1^2 \overline{z_2},\ z_1^2 \overline{z_1} - 2 z_1 z_2 \overline{z_2},\ z_2^2 \overline{z_2} - 2 z_1 z_2 \overline{z_1},\ z_2^2 \overline{z_1}\}$$

and this space decomposes into:

$$\mathcal{H}(\mathcal{P}^2(z_1, z_2) \otimes \mathcal{P}^1(\overline{z_1}, \overline{z_2})) = V_{(2,-1)} \oplus V_{(1,0)} \oplus V_{(0,1)} \oplus V_{(-1,2)}.$$

We can see that unless at least one of the conditions: $2r - s \in \mathbb{Z}, r \in \mathbb{Z}, s \in \mathbb{Z}$ or $2s - r \in \mathbb{Z}$ are satisfied, there is no fixed vector in any of the weight spaces.

Doing a similar argument, one can see that a basis in the space $\mathcal{H}(\mathcal{P}^1(z_1, z_2) \otimes \mathcal{P}^2(\overline{z_1}, \overline{z_2}))$ is given by the polynomials

$$\{z_2 \overline{z_1}^2,\ z_1 \overline{z_1}^2 - 2 z_2 \overline{z_1}\,\overline{z_2},\ z_2 \overline{z_2}^2 - 2 z_1 \overline{z_1}\,\overline{z_2},\ z_1 \overline{z_2}^2\}$$

and the space decomposes into:

$$\mathcal{H}(\mathcal{P}^1(z_1, z_2) \otimes \mathcal{P}^2(\overline{z_1}, \overline{z_2})) = V_{(-2,1)} \oplus V_{(-1,0)} \oplus V_{(0,-1)} \oplus V_{(1,-2)}.$$

Unless at least one of the conditions: $-2r+s\in\mathbb{Z}, -r\in\mathbb{Z}, -s\in\mathbb{Z}$ or $-2s+r\in\mathbb{Z}$ is satisfied, there is no fixed vector in any of the weight spaces.

Finally, a basis in the space $\mathcal{H}^3(\overline{z_1},\overline{z_2})$ is given by the polynomials $\{\overline{z_1}^3, \overline{z_1}^2\overline{z_2}, \overline{z_1}\,\overline{z_2}^2, \overline{z_2}^3\}$ and this space decomposes into the weight spaces:

$$\mathcal{H}^3(\overline{z_1},\overline{z_2}) = V_{(-3,0)} \oplus V_{(-2,-1)} \oplus V_{(-1,-2)} \oplus V_{(-0,-3)}.$$

For there to be a fixed vector in this space, at least one of the following conditions should be satisfied: $-3r\in\mathbb{Z}$, $-2r-s\in\mathbb{Z}$, $-r-2s\in\mathbb{Z}$ or $-3s\in\mathbb{Z}$.

Therefore, the space of complexified harmonic cubics decomposes under the action of the maximal torus into 8 pairs of opposite weight spaces, each of multiplicity one:

$$\begin{aligned}\mathcal{H}^3_{\mathbb{C}} = {}& V_{(3,0)} \oplus V_{(-3,0)} \oplus V_{(2,1)} \oplus V_{(-2,-1)} \oplus V_{(1,2)} \\ & \oplus V_{(-1,-2)} \oplus V_{(0,3)} \oplus V_{(0,-3)} \oplus V_{(2,-1)} \\ & \oplus V_{(-2,1)} \oplus V_{(1,0)} \oplus V_{(-1,0)} \oplus V_{(0,1)} \oplus V_{(0,-1)} \oplus V_{(1,-2)} \oplus V_{(-1,2)}.\end{aligned}$$

A real harmonic cubic is the sum of elements drawn from these weight spaces, with the coefficients in opposite weight spaces being complex conjugates. Then, there exists a fixed element in the space of real harmonic cubics in 4 variables if and only if there are nontrivial elements in the maximal torus that act trivially on at least one pair of these weight spaces. By the above analysis, one can see that this is equivalent to the satisfaction of at least one of the following conditions: (1) $3r\in\mathbb{Z}$, (2) $r\in\mathbb{Z}$, (3) $2r+s\in\mathbb{Z}$, (4) $2r-s\in\mathbb{Z}$, (5) $2s+r\in\mathbb{Z}$, (6) $2s-r\in\mathbb{Z}$, (7) $3s\in\mathbb{Z}$, (8) $s\in\mathbb{Z}$. Next we assume that exactly one of the conditions above is satisfied:

1) $3r\in\mathbb{Z}$. In this case g acts trivially on the pair of opposite weight spaces $V_{(3,0)}$ and $V_{(-3,0)}$ and the fixed real harmonic cubics in 4 variables are of the form $az_1^3+\bar{a}\overline{z_1}^3$. So,

$$C = \mathrm{Re}(az_1^3),$$

where $a\in\mathbb{C}$. Therefore, a basis in the space of fixed real harmonic cubics is given by the harmonic polynomials $\{x_1^3-3x_1x_2^2,\ 3x_1^2x_2-x_2^3\}$.

2) $r\in\mathbb{Z}$. This condition implies also $3r\in\mathbb{Z}$ and g acts trivially on the pairs of opposite weight spaces $V_{(3,0)}, V_{(-3,0)}, V_{(1,0)}$ and $V_{(-1,0)}$. So, the fixed real harmonic cubics are of the form:

$$C = \mathrm{Re}(az_1^3 + b(z_1^2\overline{z_1} - 2z_1z_2\overline{z_2})),$$

where $a, b \in \mathbb{C}$. Therefore, a basis in the space of fixed real harmonic cubics is:

$$\{x_1^3 - 3x_1x_2^2,\ 3x_1^2x_2 - x_2^3,\ x_1(x_1^2+x_2^2-2x_3^2-2x_4^2),\ x_2(x_1^2+x_2^2-2x_3^2-2x_4^2)\}.$$

3) $2r + s \in \mathbb{Z}$. Then g acts trivially on the pair of opposite weight spaces $V_{(2,1)}$ and $V_{(-2,-1)}$ and the fixed real harmonic cubics are of the form:

$$C = \mathrm{Re}(az_1^2 z_2),$$

where $a \in \mathbb{C}$. A basis in the space of fixed real harmonic cubics is given by the polynomials: $\{(x_1^2 - x_2^2)x_3 - 2x_1x_2x_4,\ (x_1^2 - x_2^2)x_4 + 2x_1x_2x_3\}$

4) $2r - s \in \mathbb{Z}$. The element g acts trivially on $V_{(2,-1)}$ and $V_{(-2,1)}$ and the fixed real harmonic cubics are of the form:

$$C = \mathrm{Re}(az_1^2 \overline{z_2}),$$

where $a \in \mathbb{C}$. Thus, a basis in the space of fixed real harmonic cubics is given by the polynomials: $\{(x_1^2 - x_2^2)x_3 + 2x_1x_2x_4,\ (x_1^2 - x_2^2)x_4 - 2x_1x_2x_3\}$

5) $2s + r \in \mathbb{Z}$. In this case g acts trivially $V_{(1,2)}$ and $V_{(-1,-2)}$ and

$$C = \mathrm{Re}(az_1 z_2^2),$$

where $a \in \mathbb{C}$ is the general harmonic cubic polynomial fixed by the action. Therefore, a basis for the space of fixed real harmonic cubics is given by the harmonic polynomials $\{(x_3^2 - x_4^2)x_1 - 2x_2x_3x_4,\ (x_3^2 - x_4^2)x_2 + 2x_1x_3x_4\}$.

6) $2s - r \in \mathbb{Z}$. Then g acts trivially on the pair of opposite weight spaces $V_{(-1,2)}$ and $V_{(1,-2)}$ and the fixed real harmonic cubics are of the form:

$$C = \mathrm{Re}(z_2^2 \overline{z_1}),$$

where $a \in \mathbb{C}$. A basis is given by the polynomials: $\{(x_3^2 - x_4^2)x_1 + 2x_2x_3x_4,\ (x_3^2 - x_4^2)x_2 - 2x_1x_3x_4\}$

7) $3s \in \mathbb{Z}$. Now g acts trivially on the pair of opposite weight spaces $V_{(0,3)}$ and $V_{(0,-3)}$ and the fixed real harmonic cubics are of the form:

$$C = \mathrm{Re}(az_2^3),$$

where $a \in \mathbb{C}$ and therefore, a basis is given by $\{x_3^3 - 3x_3x_4^2,\ 3x_3^2x_4 - x_4^3\}$.

8) $s \in \mathbb{Z}$. In this last case, g acts trivially on $V_{(0,3)}, V_{(0,-3)}, V_{(0,1)}$ and $V_{(0,-1)}$. The real harmonic cubics fixed by the action are of the form:

$$C = \mathrm{Re}(az_2^3 + b(z_2^2\overline{z_2} - 2z_1z_2\overline{z_1})),$$

where $a, b \in \mathbb{C}$. Therefore, a basis in the space of fixed real harmonic cubics is:

$$\Big\{x_3^3 - 3x_3x_4^2,\ 3x_3^2x_4 - x_4^3,\ x_3(2x_1^2 + 2x_2^2 - x_3^1 - x_4^2),\\ x_4(2x_1^2 + 2x_2^2 - x_3^2 - x_4^2)\Big\}.$$

q.e.d.

Remark 1. In Figure 1 below we graphed in the coordinates (r, s) mod $\mathbb{Z}$ all the possibilities appearing in Proposition 3.9. By moding out by the Weyl group of SO(4), we can consider the possibilities only in the triangle found by intersecting the regions below the lines $r = s$ and $s = 1 - r$. Furthermore, since the stabilizer of g in SO(4) coincides with its stabilizer in O(4), we can actually mod out by the Weyl group of O(4). The elements $\begin{pmatrix} e^{2\pi ir} & 0 \\ 0 & e^{2\pi is} \end{pmatrix}$ and $\begin{pmatrix} e^{2\pi ir} & 0 \\ 0 & e^{-2\pi is} \end{pmatrix}$ are conjugate to each other in O(4), by the element $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in \mathrm{O}(4)$. Therefore, we can restrict our attention to the cases found in the small shaded triangle shown in Figure 1, which is a fundamental Weyl chamber.

Remark 2. If only one of the conditions in the small triangle are satisfied, the stabilizer G will be at least an SO(2), thus continuous, and we will recover the cases already studied in Section 3.2. For example, if only $2s - r \in \mathbb{Z}$, moding out by $\mathbb{Z}$, we get $2s - r = 0$. The stabilizer group in this case looks like $\left\{\begin{pmatrix} e^{2i\theta} & 0 \\ 0 & e^{i\theta} \end{pmatrix}, \theta \in \mathbb{R}\right\}$ and this is the case of continuous symmetry studied in Section 3.2.3.

Therefore, in order to find the fundamental cubics that have discrete nontrivial stabilizers under the action of SO(4), we have to look at elements that have nontrivial components in at least two non-opposite weight spaces. As seen in Figure 1, up to conjugacy in O(4), there are six nontrivial elements in the maximal torus that act trivially on more than two pairs of weight spaces.

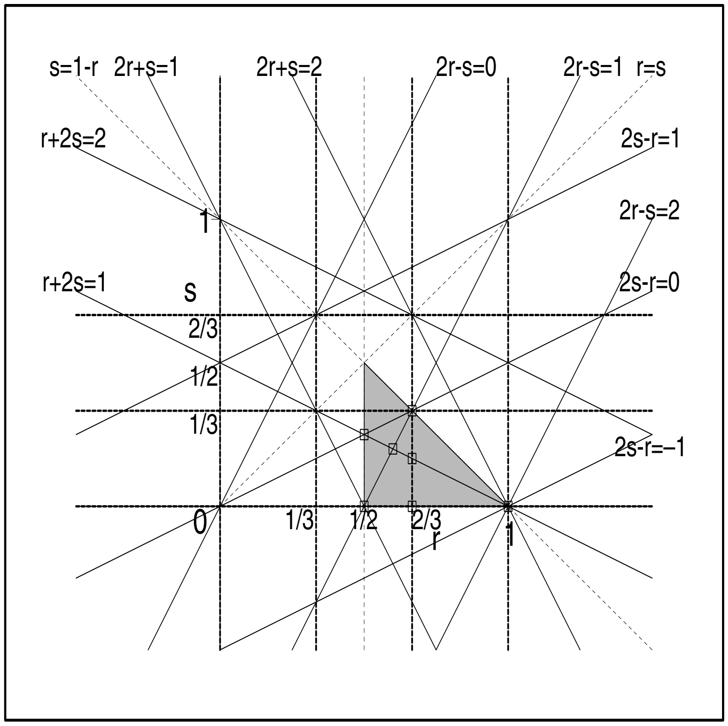

Figure 1. The weight spaces and a fundamental Weyl chamber.

Corollary 3.10. *If G is a nontrivial discrete subgroup of* SO(4) *that stabilizes a nontrivial polynomial $h \in \mathcal{H}_3(\mathbb{R}^4)$, then G can not have elements of order > 6.*

Proof. This follows from Proposition 3.9 and the above remarks. Any element in G is conjugate to an element of the form $g = \begin{pmatrix} e^{2\pi i r} & 0 \\ 0 & e^{2\pi i s} \end{pmatrix}$, where $r = \frac{m}{p} \in \mathbb{Q}$, $s = \frac{n}{q} \in \mathbb{Q}$, $r, s < 1$.

By looking at the small triangle in Figure 1, we can see that there are the following possibilities for the values of r and s mod $\mathbb{Z}$ and mod the Weyl group:

1) If $r + 2s \in \mathbb{Z}$ and $3r \in \mathbb{Z}$, then $r = \frac{2}{3}$ and $s = \frac{1}{6}$. The element $g = \begin{pmatrix} e^{\frac{4\pi i}{3}} & 0 \\ 0 & e^{\frac{\pi i}{3}} \end{pmatrix}$ has **order 6** and acts trivially on the pairs of opposite weight spaces $V_{(3,0)}, V_{(-3,0)}$ and $V_{(1,2)}, V_{(-1,-2)}$. The general harmonic cubic stabilized by this element is

$$C = \mathrm{Re}(a z_1^3 + b z_1 z_2^2),\ a, b \in \mathbb{C}.$$

2) If $r+2s \in \mathbb{Z}$ and $2r-s \in \mathbb{Z}$, then we get $r=\frac{3}{5}$ and $s=\frac{1}{5}$. The element $g=\begin{pmatrix} e^{\frac{6\pi i}{5}} & 0 \\ 0 & e^{\frac{2\pi i}{5}} \end{pmatrix}$ has **order 5** and acts trivially on the pairs of opposite weight spaces $V_{(1,2)}, V_{(-1,-2)}$ and $V_{(2,-1)}, V_{(-2,1)}$. Therefore, the general harmonic cubic stabilized by this element is

$$C = \mathrm{Re}(az_1 z_2^2 + bz_1^2 \overline{z_2}), \ a,b \in \mathbb{C}.$$

3) If $2s-r \in \mathbb{Z}$ and $r+2s \in \mathbb{Z}$, then we get $(r,s)=(\frac{1}{2},\frac{1}{4})$. The element $g=\begin{pmatrix} -1 & 0 \\ 0 & i \end{pmatrix}$ has **order 4** and acts trivially on the pairs of opposite weight spaces $V_{(-1,2)}, V_{(1,-2)}$ and $V_{(1,2)}, V_{(-1,-2)}$. The general harmonic cubic stabilized by this element is

$$C = \mathrm{Re}(a\overline{z_1} z_2^2 + bz_1 z_2^2), \ a,b \in \mathbb{C}.$$

4) If $s \in \mathbb{Z}$ and $3r \in \mathbb{Z}$, then we get $(r,s)=(\frac{2}{3},0)$. The element $g=\begin{pmatrix} e^{\frac{4\pi i}{3}} & 0 \\ 0 & I_2 \end{pmatrix}$ has **order 3** and acts trivially on the pairs of opposite weight spaces $V_{(3,0)}, V_{(-3,0)}, V_{(0,1)}, V_{(0,-1)}$ and $V_{(0,3)}, V_{(0,-3)}$. The general harmonic cubic fixed by this element is

$$C = \mathrm{Re}(az_1^3 + bz_2^3 + c(z_2^2\overline{z_2} - 2z_1\overline{z_1}z_2)), \ a,b,c \in \mathbb{C}.$$

5) If $2s-r \in \mathbb{Z}$, $2r-s \in \mathbb{Z}$, $3r \in \mathbb{Z}$ and $3s \in \mathbb{Z}$ then we get $(r,s)=(\frac{2}{3},\frac{1}{3})$. The element $g=\begin{pmatrix} e^{\frac{4\pi i}{3}} & 0 \\ 0 & e^{\frac{2\pi i}{3}} \end{pmatrix}$ has **order 3** and acts trivially on the pairs of opposite weight spaces $V_{(-1,2)}, V_{(1,-2)}, V_{(2,-1)}, V_{(-2,1)}, V_{(3,0)}, V_{(-3,0)}$ and $V_{(0,3)}, V_{(0,-3)}$. The general harmonic cubic stabilized by this element is

$$C = \mathrm{Re}(a\overline{z_1}z_2^2 + bz_1^2\overline{z_2} + cz_1^3 + ez_2^3), \ a,b,c,e \in \mathbb{C}.$$

6) If $s \in \mathbb{Z}$, $2r+s \in \mathbb{Z}$ and $2r-s \in \mathbb{Z}$, then we get $(r,s)=(\frac{1}{2},0)$. The element $g=\begin{pmatrix} -I_2 & 0 \\ 0 & I_2 \end{pmatrix}$ has **order 2** and acts trivially on the pairs of opposite weight spaces $V_{(0,1)}, V_{(0,-1)}, V_{(2,1)}, V_{(-2,-1)}$,
$V_{(2,-1)}, V_{(-2,1)}$ and $V_{(0,3)}, V_{(0,-3)}$. The general harmonic cubic stabilized by this element is

$$C = \mathrm{Re}(az_2^3 + b(z_2^2\overline{z_2} - 2z_1\overline{z_1}z_2) + cz_1^2 z_2 + ez_1^2\overline{z_2}), \ a,b,c,e \in \mathbb{C}.$$

7) If $r \in \mathbb{Z}$, $s \in \mathbb{Z}$, $r+2s \in \mathbb{Z}$, $2r+s \in \mathbb{Z}$, $2s-r \in \mathbb{Z}$ and $2r-s \in \mathbb{Z}$, meaning all the conditions are satisfied at once, then we get $r=1$ and $s=0$, so g is just the **identity** element.

3.3.1 Polyhedral symmetry

Now we are going to find the nontrivial harmonic cubic polynomials in 4 variables whose stabilizer is one of the polyhedral subgroups of SO(4) described in Section 3.1. and we will study the families of special Lagrangian 4-folds with this stabilizer type.

Proposition 3.11. *The* SO(4)*-stabilizer of* $C \in \mathcal{H}_3(\mathbb{R}^4)$ *is a polyhedral subgroup of* SO(4) *if and only if* C *lies on the* SO(4)*-orbit of exactly one of the following polynomials:*

1. $rx_1(x_1^2 - x_2^2 - x_3^2 - x_4^2) + sx_2x_3x_4$, *for some* $r, s > 0$ *satisfying* $s \neq 2\sqrt{5}r$, *whose stabilizer is the tetrahedral subgroup* $\mathbb{T}$ *of* SO(4);
2. $sx_2x_3x_4$, *for some* $s > 0$, *whose stabilizer is the irreducibly acting octahedral subgroup* $\mathbb{O}^+$;
3. $r[x_1(x_1^2 - x_2^2 - x_3^2 - x_4^2) + 2\sqrt{5}x_2x_3x_4]$, $r > 0$, *whose stabilizer is the irreducibly acting icosahedral subgroup* $\mathbb{I}^+$.

Proof. The polyhedral subgroups of SO(4) were found to be the tetrahedral subgroup $\mathbb{T}$ of order 12, the reducibly and irreducibly acting octahedral subgroups $\mathbb{O}$ and $\mathbb{O}^+$, each of order 24, the reducibly and irreducibly acting icosahedral subgroups $\mathbb{I}$ and $\mathbb{I}^+$, each of order 60.

First we look at the tetrahedral subgroup $\mathbb{T}$ and find the harmonic cubics in 4 variables $\{x_1, x_2, x_3, x_4\}$ that are stabilized by this subgroup. As we have seen in Section 3.1, $\mathbb{T} = \{[t, t] | t \in \mathbf{T}\}$, where $\mathbf{T} = \{\pm\mathbf{1}, \pm\mathbf{i}, \pm\mathbf{j}, \pm\mathbf{k}, \frac{1}{2}(\pm\mathbf{1} \pm \mathbf{i} \pm \mathbf{j} \pm \mathbf{k})\}$ is the binary tetrahedral subgroup of the unit quaternion group U, of order 24. The subgroup $\mathbb{T}$ sits in SO(3) and it is generated by the transformations: $[\mathbf{i}, \mathbf{i}]$ with representing matrix $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$, relative to the basis $\{\mathbf{1}, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$, $[\mathbf{j}, \mathbf{j}]$ with representing matrix $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$ and $[\frac{1}{2}(\mathbf{1} + \mathbf{i} + \mathbf{j} + \mathbf{k}), \frac{1}{2}(\mathbf{1} + \mathbf{i} + \mathbf{j} + \mathbf{k})]$ with representing matrix $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$. We can see that $\mathbb{T}$ fixes a cubic in $\{x_1, x_2, x_3, x_4\}$ if and only if the cubic is invariant under the flips of the signs of two of the coordinates $\{x_2, x_3, x_4\}$ and also under permuting $\{x_2, x_3, x_4\}$ while keeping x_1 fixed. Therefore, the cubic should be a linear combination of the polynomials x_1^3, $x_1(x_2^2 + x_3^2 + x_4^2)$ and $x_2x_3x_4$. Now, considering the extra condition that the cubic should be harmonic, it follows that the harmonic cubics stabilized by $\mathbb{T}$ lie on the SO(4)-orbit of the polynomial

$$C = rx_1(x_1^2 - x_2^2 - x_3^2 - x_4^2) + sx_2x_3x_4, \tag{3.32}$$

for some $r, s \geq 0$.

We turn now to the harmonic cubics invariant under the reducibly acting octahedral subgroup $\mathbb{O}$, which sits in SO(3). As we have seen in Section 3.1, the group $\mathbb{O} = \{[o, o] \mid o \in \mathbf{O}\}$, where $\mathbf{O} = \mathbf{T} \cup \frac{1}{\sqrt{2}}(\mathbf{1}+\mathbf{i})\mathbf{T}$ is the octahedral binary subgroup of U, of order 48. Since $\mathbb{O}$ contains $\mathbb{T}$, it follows that $\mathbb{O}$ is generated by the generators of $\mathbb{T}$ and the extra element $\left[\frac{1}{\sqrt{2}}(\mathbf{1}+\mathbf{i}), \frac{1}{\sqrt{2}}(\mathbf{1}+\mathbf{i})\right]$ with representing matrix $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$. This extra element fixes the polynomial (3.32) if and only if $s = 0$. Therefore, the harmonic cubics stabilized by $\mathbb{O}$ lie on the SO(4)-orbit of the polynomial $rx_1(x_1^2 - x_2^2 - x_3^2 - x_4^2), r > 0$ which has full symmetry SO(3).

We look now for harmonic cubics invariant under the irreducibly acting octahedral subgroup $\mathbb{O}^+ = \{[o, o], o \in \mathbf{T}$ and $[o, -o], o \in \frac{1}{\sqrt{2}}(\mathbf{1} + \mathbf{i})\mathbf{T}\}$. The subgroup $\mathbb{O}^+$ contains $\mathbb{T}$ and it is generated by the generators of $\mathbb{T}$ plus the extra element $\left[\frac{1}{\sqrt{2}}(\mathbf{1}+\mathbf{i}), -\frac{1}{\sqrt{2}}(\mathbf{1}+\mathbf{i})\right]$. This extra element fixes the harmonic polynomial (3.32) if and only if $r = 0$. Therefore, the harmonic cubics stabilized by $\mathbb{O}^+$ lie on the SO(4)-orbit of the polynomial $sx_2x_3x_4,\ s > 0$.

We look now for the harmonic cubics invariant under the reducibly acting icosahedral subgroup $\mathbb{I}$, which sits in SO(3). As we have seen in Section 3.1, $\mathbb{I} = \{[l, l] \mid l \in \mathbf{I}\}$, where $\mathbf{I} = \cup_{k=0}^{4}(\frac{1}{2\tau} + \frac{\tau}{2}\mathbf{i} + \frac{1}{2}\mathbf{j})^k\mathbf{T}$ is the binary icosahedral subgroup of U, of order 120 and $\tau = \frac{\sqrt{5}+1}{2}$. The subgroup $\mathbb{I}$ contains $\mathbb{T}$ and it is generated by the generators of $\mathbb{T}$ plus the extra element $\left[\frac{1}{2\tau} + \frac{\tau}{2}\mathbf{i} + \frac{1}{2}\mathbf{j}, \frac{1}{2\tau} + \frac{\tau}{2}\mathbf{i} + \frac{1}{2}\mathbf{j}\right]$. Straightforward calculations show that this extra element fixes the harmonic polynomial (3.32) if and only if $s = 0$. Therefore, the harmonic cubics stabilized by $\mathbb{I}$ lie on the SO(4)-orbit of the polynomial $rx_1(x_1^2 - x_2^2 - x_3^2 - x_4^2)$ which has full symmetry SO(3).

Finally, we look for the harmonic cubics invariant under the irreducibly acting icosahedral subgroup $\mathbb{I}^+$. From Section 3.1, $\mathbb{I}^+ = \{[r^+, r] \mid r \in \mathbf{I}\}$, where r^+ is the image of $r \in \mathbf{I}$ under the automorphism of the quaternion field that changes the sign of $\sqrt{5}$. This automorphism exchanges τ for $-\frac{1}{\tau}$. The subgroup $\mathbb{I}^+$ contains $\mathbb{T}$ and it is generated by the generators of $\mathbb{T}$ plus the extra element $\left[(\frac{1}{2\tau}+\frac{\tau}{2}\mathbf{i}+\frac{1}{2}\mathbf{j})^+, \frac{1}{2\tau}+\frac{\tau}{2}\mathbf{i}+\frac{1}{2}\mathbf{j}\right] = \left[-\frac{\tau}{2} - \frac{1}{2\tau}\mathbf{i} + \frac{1}{2}\mathbf{j}, \frac{1}{2\tau} + \frac{\tau}{2}\mathbf{i} + \frac{1}{2}\mathbf{j}\right]$. Straightforward calculations show that this extra element fixes the harmonic polynomial (3.32) if and only if

$s = 2\sqrt{5}r$. Therefore, the harmonic cubics stabilized by $\mathbb{I}^+$ lie on the SO(4)-orbit of the polynomial $r[x_1(x_1^2 - x_2^2 - x_3^2 - x_4^2) + 2\sqrt{5}x_2x_3x_4]$.

To conclude, one can easily compute that the identity component of the stabilizer of the polynomial (3.32) is always discrete, except in the case $s = 0$. q.e.d.

We now consider those special Lagrangian submanifolds $L \subset \mathbb{C}^4$ whose cubic fundamental form has a polyhedral symmetry at each point.

Theorem 3.12. *Suppose that $L \subset \mathbb{C}^4$ is a connected special Lagrangian 4-fold with the property that its fundamental cubic at each point has a tetrahedral symmetry $\mathbb{T}$. Then, up to congruence and scaling, L is the Harvey-Lawson example $L \subset \mathbb{C}^4$ defined in standard coordinates by the equations*

$$\begin{aligned} L: \quad & |z_0| = |z_1| = |z_2| = |z_3| \\ & \mathrm{Re}(z_0z_1z_2z_3) = \sqrt[5]{2}. \end{aligned}$$

Proof. Let L be a special Lagrangian 4-fold that satisfies the hypotheses of the theorem and let C be its fundamental cubic. Proposition 3.11 implies that there exist functions $r, s \colon L \to \mathbb{R}_+$ with $s \neq \sqrt{5}r$ and a $\mathbb{T}$-subbundle $F \subset P_L$ over L for which the following identity holds:

$$C = 3r\omega_1(\omega_1^2 - \omega_2^2 - \omega_3^2 - \omega_4^2 -) + 6s\omega_2\omega_3\omega_4$$

Since F is an $\mathbb{T}$-bundle, the following relations hold: $\alpha_{21} = \alpha_{31} = \alpha_{41} = \alpha_{32} = \alpha_{42} = \alpha_{43} = 0 \mod \{\omega_1, \omega_2, \omega_3, \omega_4\}$. The usual differential analysis yields the following structure equations on F:

$$\begin{aligned} d\omega_1 &= 0 \\ d\omega_2 &= \sqrt{s^2 - r^2}\omega_1 \wedge \omega_2 \\ d\omega_3 &= \sqrt{s^2 - r^2}\omega_1 \wedge \omega_3 \\ d\omega_4 &= \sqrt{s^2 - r^2}\omega_1 \wedge \omega_4 \\ dr &= -5r\sqrt{s^2 - r^2}\omega_1 \\ ds &= -s\sqrt{s^2 - r^2}\omega_1. \end{aligned} \tag{3.33}$$

From the last two equations in (3.33), it follows that $r = cs^5$, $c > 0$ constant. We can suppose that $c = 1$ since the equations are invariant under scaling. Moreover, $s \in [-1, 1]$.

The above structure equations imply that $\omega_1 = 0$ defines an integrable 3-plane field which we denote by Γ_2 and that $\omega_2 = \omega_3 = \omega_4 = 0$ defines an integrable 1-plane field denoted by Γ_1. Since $d\omega_1 = 0$, it follows that $\omega_1 = dx_1$ on the leaves of the foliation Γ_1. The structure equations (3.33) also imply $d(s\omega_2) = d(s\omega_3) = d(s\omega_4) = 0$ and therefore there exist functions x_2, x_3, x_4 on L such that $\omega_2 = \frac{dx_2}{s}$, $\omega_3 = \frac{dx_3}{s}$ and $\omega_4 = \frac{dx_4}{s}$. The metric $g = \frac{dx_2^2+dx_3^2+dx_4^2}{s^2}$ is well-defined on the leaves of the Γ_2 foliation.

Equations $de_i = e_j\alpha_{ji} - Je_j\beta_{ji}$ and $d(Je_i) = e_j\beta_{ji} + Je_j\alpha_{ji}$ yield that, as matrices

$$d(e_1 \quad Je_1) = (e_1 \quad Je_1)\begin{pmatrix} 0 & 3s^5\omega_1 \\ -3s^5\omega_1 & 0 \end{pmatrix} \mod \{\omega_2, \omega_3, \omega_4\}.$$

Therefore, the leaves of the Γ_1 foliation are plane curves with curvature $k = 3s^5$, lying in the complex line (e_1, Je_1). These curves are congruent since ds is a multiple of ω_1.

Now consider the Γ_2 foliation, defined by the equation $\omega_1 = 0$. Since s is constant on its leaves, the connection matrix $A = \begin{pmatrix} \alpha_{ij} & \beta_{ij} \\ -\beta_{ij} & \alpha_{ij} \end{pmatrix}$ satisfies $A \wedge A = dA = 0$. Therefore A takes values in a 3-dimensional abelian subalgebra $\mathfrak{g} \subset \mathfrak{su}(4)$. The maximal torus of SU(4) is conjugate to the subgroup

$$T^3 = \left\{ \operatorname{diag}(e^{i\theta_0}, e^{i\theta_1}, e^{i\theta_2}, e^{i\theta_3}) \mid \sum_{k=0}^{3} \theta_i = 0 \mod 2\pi \right\}$$

and the maximal torus acts on L by rotating around a plane curve C. Therefore, the solution is invariant under the torus action and the only special Lagrangian 4-folds with this property are described explicitly by Harvey and Lawson in their paper [12]. If (z_0, z_1, z_2, z_3) are coordinates on $\mathbb{C}^4$, then the special Lagrangian 4-folds in $\mathbb{C}^4$ invariant under T^3 look like:

$$\begin{gathered} |z_0|^2 - |z_1|^2 = c_1, \quad |z_0|^2 - |z_2|^2 = c_2, \quad |z_0|^2 - |z_3|^2 = c_3, \\ \operatorname{Re}(z_0 z_1 z_2 z_3) = a \end{gathered} \tag{3.34}$$

for some real constants a, c_1, c_2, c_3. It is easy to see that the solution of the structure equations (3.33) is symmetric in (z_1, z_2, z_3), therefore $c_1 = c_2 = c_3 = c$.

Reparametrizing the solution using polar coordinates $z_k = r_k e^{i\theta_k}$, $k = 0...3$, (3.34) becomes

$$r_0^2 - r_1^2 = r_0^2 - r_2^2 = r_0^2 - r_3^2 = c$$
$$\theta_0 = \arccos \frac{a}{r_0 r_1 r_2 r_3} - \theta_1 - \theta_2 - \theta_3. \tag{3.35}$$

We will find out for what constants a and c, the special Lagrangian 4-fold defined by (3.35) is a solution of the structure equations. As we have seen, the solution is a special Lagrangian 4-fold which is foliated by congruent curves of curvature $3s^5$ and 3-manifolds which are the T^3-orbit of the points on the leaves of the first foliation.

Since $z(r, \theta_1, \theta_2, \theta_3) = (\sqrt{c + r^2}, \arccos \frac{a}{r^3\sqrt{c+r^2}} - \theta_1 - \theta_2 - \theta_3, r, \theta_1, r, \theta_2, r, \theta_3)$, the tangent plane to a T^3-orbits is spanned by the vectors $v_1 = z_{\theta_1} = -\frac{\partial}{\partial\theta_0} + \frac{\partial}{\partial\theta_1}$, $v_2 = z_{\theta_2} = -\frac{\partial}{\partial\theta_0} + \frac{\partial}{\partial\theta_2}$ and $v_3 = z_{\theta_3} = -\frac{\partial}{\partial\theta_0} + \frac{\partial}{\partial\theta_3}$.

We look now for another vector v_0 in the tangent space of L such that $\{v_0, v_1, v_2, v_3\}$ are a basis of this tangent space. Since this tangent space is special Lagrangian, the symplectic form ω and the imaginary part of the holomorphic volume form Ω should vanish on it. Also, v_0 should be orthogonal to v_i, $i = 1...3$, so $g(v_0, v_i) = 0$ for $i = 1...3$.

Let us write

$$v_0 = \sum_{i=0}^{3} \mu_i \frac{\partial}{\partial\theta_i} + \nu_i \frac{\partial}{\partial r_i}.$$

The symplectic form in polar coordinates is $\omega = \sum_{i=0}^{3} r_i dr_i \wedge d\theta_i$ and the condition $\omega(v_0, v_i) = 0$ for $i = 1, 2, 3$ implies that $\nu_i = \frac{r_0\nu_0}{r_i} = \frac{\nu}{r_i}$, where $\nu = r_0\nu_0$. The metric on $\mathbb{C}^4$ is $g = \sum_{i=0}^{3}(dr_i)^2 + r_i^2(d\theta_i)^2$ and the condition that $g(v_0, v_i) = 0$ for $i = 1...3$ implies the relations $\mu_i = \frac{\mu_0 r_0^2}{r_i^2} = \frac{\mu}{r_i^2}$, for $i = 1, 2, 3$, where $\mu = \mu_0 r_0^2$. Finally, straightforward calculations yield that the condition $\operatorname{Im}\Omega(v_1, v_2, v_3, v_0) = 0$ implies the relation $\frac{\nu}{\mu} = \tan(\theta_0 + \theta_1 + \theta_2 + \theta_3)$. Therefore,

$$v_0 = \sum_{i=0}^{3} \frac{1}{r_i^2}\frac{\partial}{\partial\theta_i} + \frac{\tan(\theta_0 + \theta_1 + \theta_2 + \theta_3)}{r_i}\frac{\partial}{\partial r_i}.$$

Next, we find an integral curve of the vector field v_0, that lies in:

$$L: \quad r_1 = r_2 = r_3 = r, \quad r_0 = \sqrt{c + r^2},$$
$$\theta_0 = \arccos \frac{a}{r^3\sqrt{c + r^2}} - \theta_1 - \theta_2 - \theta_3.$$

When $c = 0$, an integral curve is given by

$$C: \quad r_0 = r_1 = r_2 = r_3 = r,\ \theta_0 = \theta_1 = \theta_2 = \theta_3 = \theta,\ \cos(4\theta) = \frac{a}{r^4}.$$

Therefore, the curve C is given by: $z_1 = z_2 = z_3 = z_4 = re^{i\theta}$, $r^4\cos(4\theta) = a$ and it is a plane curve which lies in the complex line $z_1 = z_2 = z_3 = z_4$. We have seen that L is foliated by plane curves with curvature $3s^5$, so we will determine for what value of a the curve C has this curvature. We choose an orthonormal basis in the $z_1 = z_2 = z_3 = z_4$ plane: $\mathbf{e_1} = (\frac{1}{2}, 0, \frac{1}{2}, 0, \frac{1}{2}, 0, \frac{1}{2}, 0)$ and $\mathbf{e_2} = (0, \frac{1}{2}, 0, \frac{1}{2}, 0, \frac{1}{2}, 0, \frac{1}{2})$ and in this basis, the curve C is given by $\gamma(\theta) = (2r\cos\theta, 2r\sin\theta)$, where $r = \left(\frac{a}{\cos(4\theta)}\right)^{\frac{1}{4}}$.

Computing the curvature of γ, one gets $k(\theta) = -\frac{3}{2}a^{-\frac{1}{4}}(\cos(4\theta))^{\frac{5}{4}}$. But the curves that foliate L are parameterized by arclength

$$\omega_1 = dt = d\theta|\gamma\,'| = 2a^{\frac{1}{4}}(\cos\theta)^{-\frac{5}{4}} \tag{3.36}$$

and the curvature in this parameterization is $k = 3s^5$. Therefore

$$s = (\frac{k}{3})^{\frac{1}{5}} = -\frac{1}{2^{\frac{1}{5}}a^{\frac{1}{20}}}(\cos 4\theta)^{\frac{1}{4}}. \tag{3.37}$$

From the structure equations (3.33), it follows that $ds = -s\sqrt{s^2 - s^{10}}\omega_1$ has to be satisfied. Using equations (3.36) and (3.37), we get

$$ds = \frac{1}{2^{\frac{6}{5}}a^{\frac{3}{10}}}(\cos 4\theta)^{\frac{1}{2}}\sin 4\theta\ \omega_1 \tag{3.38}$$

and from equation (3.38),

$$\begin{aligned} &- s\sqrt{s^2 - s^{10}}\omega_1 \\ &\quad = \frac{1}{2^{\frac{1}{5}}a^{\frac{1}{20}}}(\cos 4\theta)^{\frac{1}{4}}\left(\frac{1}{2^{\frac{2}{5}}a^{\frac{1}{10}}}(\cos 4\theta)^{\frac{1}{2}} - \frac{1}{4a^{\frac{1}{2}}}(\cos 4\theta)^{\frac{5}{2}}\right)^{\frac{1}{2}}\omega_1. \end{aligned} \tag{3.39}$$

Equating these last two equations, it follows that $a = \sqrt[5]{2}$. The structure equations are satisfied now and L is a special Lagrangian 4-fold.

To conclude, the special Lagrangian submanifold L that is a solution to the structure equations (3.33) can be described explicitly as:

$$L: \quad |z_0| = |z_1| = |z_2| = |z_3|, \quad \mathrm{Re}(z_0z_1z_2z_3) = \sqrt[5]{2}.$$

q.e.d.

Theorem 3.13. *Suppose that $L \subset \mathbb{C}^4$ is a connected special Lagrangian 4-fold with the property that its fundamental cubic at each point has an octahedral symmetry $\mathbb{O}^+$, where $\mathbb{O}^+$ is the irreducibly acting octahedral subgroup of* $\mathrm{SO}(4)$. *Then, up to congruence, L is the Harvey-Lawson cone in $\mathbb{C}^4$ defined in standard coordinates (z_0, z_1, z_2, z_3) by the equation*

$$L: \quad |z_0| = |z_1| = |z_2| = |z_3|, \quad \mathrm{Re}(z_0 z_1 z_2 z_3) = 0. \tag{3.40}$$

Proof. Let L be a special Lagrangian 4-fold that satisfies the hypotheses of the theorem and let C be its fundamental cubic. Proposition 3.11 implies that there exists a function $s\colon L \to \mathbb{R}_+$ and an $\mathbb{O}^+$-subbundle $F \subset P_L$ over L for which the following identity holds:

$$C = 6s\omega_2\omega_3\omega_4$$

and the 1-forms $\omega_1, \omega_2, \omega_3, \omega_4$ form a basis on F.

Straightforward calculations show that the structure equations are:

$$\begin{aligned} d\omega_1 = 0, \quad d\omega_2 = s\omega_1 \wedge \omega_2, \quad d\omega_3 = s\omega_1 \wedge \omega_3,\\ d\omega_4 = s\omega_1 \wedge \omega_4, \quad ds = -s^2\omega_1. \end{aligned} \tag{3.41}$$

The structure equations imply the equation

$$de_1 = s(e_2\omega_2 + e_3\omega_3 + e_4\omega_4) = s(dx - e_1\omega_1),$$

where $x\colon L^+ \to \mathbb{C}^4$. From here and the last equation in (3.41), it follows that $x = \frac{e_1}{s} + x_0$, where x_0 is a constant which we can reduce to 0 by translation. Therefore $x = \frac{e_1}{s}$. On the leaves of the foliation $\omega_2 = \omega_3 = \omega_4 = 0$, $de_1 = 0$ and thus the vector e_1 is constant along these leaves. This tells us that the special Lagrangian 4-fold L^+ is a cone on some 3-dimensional manifold $\Sigma \subset S^7$. We have to determine now for what 3-dimensional manifolds $\Sigma \subset S^7$, the cone $C(\Sigma)$ is special Lagrangian and satisfies the structure equations.

In the case $t = -s$ we obtain $x = -\frac{e_1}{s}$ and the solution is again a cone through the origin, call it L^-. We have that $L = L^+ \cup L^-$.

The connection matrix $A = \begin{pmatrix} \alpha_{ij} & \beta_{ij} \\ -\beta_{ij} & \alpha_{ij} \end{pmatrix}$ satisfies $A \wedge A = dA = 0$. Therefore A takes values in an abelian subalgebra $\mathfrak{g} \subset \mathfrak{su}(4)$. The group $G = \exp \mathfrak{g}$ is a maximal torus of $\mathrm{SU}(4)$ and it is conjugate to the diagonal torus $T^3 = \left\{ \mathrm{diag}(e^{i\theta_0}, e^{i\theta_1}, e^{i\theta_2}, e^{i\theta_3}) : \sum_{k=0}^{3} \theta_i = 0 \mod 2\pi \right\}$.

G acts transitively on the cone, so the cone is homogeneous and we have to determine which of the orbits on the 7-sphere are special Lagrangian. The solution is invariant under the torus action and therefore the links of these special Lagrangian cones are 3-dimensional tori on S^7. They are described explicitly by Harvey and Lawson in their paper [12]. It follows that L is given in standard coordinates (z_0, z_1, z_2, z_3) by (3.40).

Therefore, the special Lagrangian cone L is a union of two cones L^+ (obtained in the case $t = s$) and L^- (obtained in the case $t = -s$) with vertices at the origin through the 3-dimensional tori T^+ and T^- on S^7 given by

$$T^+ = \left\{ \left(\frac{1}{2}e^{i\theta_0}, \frac{1}{2}e^{i\theta_1}, \frac{1}{2}e^{i\theta_2}, \frac{1}{2}e^{i\theta_3} \right) : \theta_0 + \theta_1 + \theta_2 + \theta_3 = \frac{\pi}{2} \right\}$$
$$T^- = \left\{ \left(\frac{1}{2}e^{i\theta_0}, \frac{1}{2}e^{i\theta_1}, \frac{1}{2}e^{i\theta_2}, \frac{1}{2}e^{i\theta_3} \right) : \theta_0 + \theta_1 + \theta_2 + \theta_3 = \frac{3\pi}{2} \right\}.$$

Theorem 3.14. *There are no connected special Lagrangian* 4*-folds whose fundamental cubic at each point has an icosahedral symmetry* $\mathbb{I}^+$*, where* $\mathbb{I}^+$ *is the irreducibly acting icosahedral subgroup of* $\mathrm{SO}(4)$*.*

Proof. Let L be a special Lagrangian 4-fold that satisfies the hypotheses of the theorem and let C be its fundamental cubic. Proposition 3.11 implies that there exists a function $r\colon L \to \mathbb{R}_+$ for which the equation

$$C = 3r[\omega_1(\omega_1^2 - \omega_2^2 - \omega_3^2 - \omega_4^2) + 2\sqrt{5}\omega_2\omega_3\omega_4]$$

defines an $\mathbb{I}^+$-subbundle $F \subset P_L$ of the L-adapted coframe bundle $P_L \to L$. The usual differential analysis on the subbundle F yields $r = 0$, contrary to the hypothesis. q.e.d.

3.3.2 Symmetries of order 6, 5 and 4

We have seen in Corollary 3.10 that the elements of a discrete stabilizer of a fundamental cubic of a special Lagrangian 4-fold have order less or equal to 6. From the proof of this corollary, the general harmonic cubic stabilized by an element of order 6 is

$$C = \mathrm{Re}(rz_1^3 + sz_1z_2^2) = r(x_1^3 - 3x_1x_2^2) + s[(x_3^2 - x_4^2)x_1 - 2x_2x_3x_4]$$

where we can arrange r, s to be real and nonnegative by making rotations in the z_1 and in the z_2-lines. Easy computations show that the full stabilizer of C is the dihedral group on 6 elements $\mathbf{D_6}$, if $r \neq 0$ and $s \neq 0$. A similar differential analysis as in the previous cases yields the following result:

Theorem 3.15. *There are no nontrivial special Lagrangian submanifolds in $\mathbb{C}^4$ whose fundamental cubic has a discrete stabilizer which contains an element of order 6.*

Next, the general harmonic cubic stabilized by an element of order 5 is

$$C = \mathrm{Re}(r z_1 z_2^2 + s z_1^2 \overline{z_2})$$

where we can arrange $r, s \geq 0$. The same kind of analysis gives:

Theorem 3.16. *There are no nontrivial special Lagrangian submanifolds in $\mathbb{C}^4$ whose fundamental cubic has a discrete stabilizer which contains at least an element of order 5.*

Remark. From this theorem, the result in Theorem 3.14 follows immediately, since the irreducibly acting icosahedral subgroup of $\mathrm{SO}(4)$ has elements of order 5.

Next, the general harmonic cubic stabilized by an element of order 4 is

$$C = \mathrm{Re}(r \overline{z_1} z_2^2 + s z_1 z_2^2)$$

where we can arrange again r, s to be real and nonnegative. The stabilizer of C is a continuous subgroup if $r = 0$ or $s = 0$, the irreducibly acting octahedral subgroup $\mathbb{O}^+$ if $r = s$ and the dihedral group $\mathbf{D_4}$ in the rest of the cases, since the element of order 2 that flips the signs of $\{x_2, x_3\}$ belongs to the stabilizer.

We obtain:

Theorem 3.17. *There is no nontrivial special Lagrangian 4-fold in $\mathbb{C}^4$ whose fundamental cubic has a $\mathbf{D_4}$-symmetry at each point.*

For the details of the calculations in the above results see [5].

3.3.3 Discrete symmetry at least $\mathbb{Z}_3$

Now we consider those special Lagrangian 4-folds $L \subset \mathbb{C}^4$ whose fundamental cubic has at least a $\mathbb{Z}_3$-symmetry at each point. We saw in the proof of Corollary 3.10 that there are two inequivalent orbits that stabilize an element of order 3. We start with:

Case 1. $(r,s) = (\frac{2}{3}, 0)$: The general harmonic cubic fixed by the element $\mathbf{g} = \begin{pmatrix} e^{\frac{4\pi i}{3}} & 0 \\ 0 & I_2 \end{pmatrix}$ in the maximal torus is:

$$C = \mathrm{Re}(rz_1^3 + tz_2^3 + sz_2(|z_2|^2 - 2|z_1|^2)),$$

where $r, t, s \in \mathbb{C}$. By rotations in the z_1-line and z_2-line, we can arrange that r, s be real and nonnegative. By writing $t = u + iv,\ u, v \in \mathbb{R}$, the cubic C becomes:

(∗)

$$C = r(x_1^3 - 3x_1x_2^2) + sx_3(x_3^3 + x_4^2 - 2x_1^2 - 2x_2^2) + u(x_3^3 - 3x_3x_4^2) + v(x_4^3 - 3x_3^2x_4)$$

where $r, u, v, s \in \mathbb{R}$ and $r, s \geq 0$.

The next lemma tells us what the full stabilizer of C is.

Lemma 3.18. *The full stabilizer of the harmonic cubic polynomial* (∗) *is:*

1) *a continuous subgroup of* $\mathrm{SO}(4)$*, if* $r = 0$ *or* $s = u = v = 0$;

2) *the dihedral subgroup* $\mathbf{D_3}$ *generated by the order* 3 *element* $\mathbf{g}$ *and the order* 2 *element that flips the signs of* $\{x_2, x_4\}$*, if* $r \neq 0, v = 0$;

3) *the dihedral subgroup* $\mathbf{D_3}$ *generated by the order* 3 *element* $\mathbf{g}$ *and the order* 2 *element that flips the signs of* $\{x_2, x_4\}$*, if* $r \neq 0, v = 0, u = 3s$;

4) *the order* 18 *normal subgroup of* $\mathbf{D}_3 \times \mathbf{D}_3$*, if* $u = v = 0$ *and* $r, s \neq 0$;

5) *the cyclic subgroup* $\mathbb{Z}_\mathbf{3}$ *generated by the order* 3 *element* $\mathbf{g}$ *if none of the above relations among the parameters* r, s, u, v *hold.*

Proof. We denoted by G be the stabilizer of the polynomial C, where $r, s \geq 0$. A simple computation shows that G is a continuous subgroup if and only if $r = 0$ or $u = v = s = 0$. Therefore, if $r \neq 0$ and $s^2 + u^2 + v^2 \neq 0$, the stabilizer G is discrete.

When $s = 0$, we can make a rotation in the (x_3, x_4)-plane and suppose also that $v = 0$. In this case, the stabilizer of C is G, the order 18 normal subgroup of $\mathbf{D}_3 \times \mathbf{D}_3$ described as follows: Let the first $\mathbf{D}_3$ be denoted by $\mathbf{D_3}^+$ and suppose it is generated by the rotation a_1 and the reflection b_1, where $a_1^3 = 1, b_1^2 = 1,\ a_1b_1a_1 = b_1$. Denote the second $\mathbf{D}_3$

by $\mathbf{D}_3{}^-$ and suppose it is generated by the rotation a_2 and the reflection b_2, where $a_2^3 = 1,\ b_2^2 = 1,\ a_2 b_2 a_2 = b_2$. Then $\mathbf{D}_3{}^+$ consists of the elements

$$\{\theta_1^+ = 1,\ \theta_2^+ = a_1,\ \theta_3^+ = a_1^2,\ r_1^+ = b_1,\ r_2^+ = a_1 b_1,\ r_3^+ = a_1^2 b_1\}$$

and $\mathbf{D}_3{}^-$ consists of the elements

$$\{\theta_1^- = 1,\ \theta_2^- = a_2,\ \theta_3^- = a_2^2,\ r_1^- = b_2,\ r_2^- = a_2 b_2,\ r_3^- = a_2^2 b_2\}.$$

The SO(4)-stabilizer of the cubic C is formed by the 18 pair elements:

$$\{(\theta_i^+, \theta_j^-), (r_i^+, r_j^-),\ i, j = 1...3\}$$

Next, if $r \neq 0$ and $s \neq 0$, the differential analysis yields the following cases:

i) If $v = 0$, the stabilizer G of the cubic C is the dihedral subgroup $\mathbf{D_3}$ generated by the order 3 element $\mathbf{g}$ and the order 2 element that flips the signs of x_2 and x_4.

ii) If $v = 0, u = 3s$, the stabilizer G of C is also the above dihedral subgroup $\mathbf{D_3}$.

iii) In the general case, when none of the above relations among the parameters r, s, u, v hold, the stabilizer of C is $\mathbb{Z}_3$.

q.e.d.

In the case of $\mathbf{D_3}$-symmetry we obtain the following partial result:

Proposition 3.19. *There is an infinite parameter family of connected special Lagrangian submanifolds in $\mathbb{C}^4$ such that the fundamental cubic at each point has a $\mathbf{D}_3$-symmetry and is of the form $(*)$, where $v = 0$ and $r, s \neq 0$. This family depends on 2 functions of one variable.*

Proof. Let L be a special Lagrangian 4-fold that satisfies the hypotheses of the theorem and let C be its fundamental cubic. It follows that

$$C = r(\omega_1^3 - 3\omega_1\omega_2^2) + u(\omega_3^3 - 3\omega_3\omega_4^2) + 3s\omega_3(\omega_3^2 + \omega_4^2 - 2\omega_1^2 - 2\omega_2^2),$$

with $r > 0, s > 0$ defines a $\mathbf{D_3}$-subbundle $F \subset P_L$ of the adapted coframe bundle $P_L \to L$. In this case, were able to write down the structure equations that hold on the bundle F, but were unable to describe completely the family of special Lagrangian submanifolds in this case. Cartan-Kähler theorem tells us that the family should depend on 2 functions of one variables. For more details see [5]. q.e.d.

Theorem 3.20. *Let L be a connected special Lagrangian submanifolds in $\mathbb{C}^4$ such that its fundamental cubic at each point has a $\mathbf{D}_3$-symmetry and it is of the form $(*)$, where $v = 0, u = 3s$ and $r, s \neq 0$. Then L is, up to rigid motion, an open subset of the asymptotically conical special Lagrangian 4-fold given by:*

$$L_\Sigma = \{(a+ib)\mathbf{u} |\ \mathbf{u} \in \Sigma,\ \mathrm{Re}(a+ib)^4 = c\}, \tag{3.42}$$

where c is a real constant and $\Sigma \subset S^7$ is a 3-manifold with the property that the cone on it is special Lagrangian, with phase i.

Proof. Let L be a special Lagrangian 4-fold that satisfies the hypotheses of the theorem and let C be its fundamental cubic. It follows that

$$C = r(\omega_1^3 - 3\omega_1\omega_2^2) + 3s\omega_3(\omega_3^2 - \omega_1^2 - \omega_2^2 - \omega_4^2),$$

with $r > 0, s \neq 0$ defines a $\mathbf{D_3}$-subbundle $F \subset P_L$ of the adapted coframe bundle $P_L \to L$.

The structure equations on this bundle are computed to be:

(3.43)

$$\begin{aligned}
d\omega_1 &= t_1\omega_3 \wedge \omega_1 - t_2\omega_4 \wedge \omega_1 - 2t_5\omega_4 \wedge \omega_2 + t_3\omega_1 \wedge \omega_2 \\
d\omega_2 &= t_4\omega_1 \wedge \omega_2 - t_1\omega_2 \wedge \omega_3 + t_2\omega_2 \wedge \omega_4 + 2t_5\omega_4 \wedge \omega_1 \\
d\omega_3 &= 0 \\
d\omega_4 &= 6t_5\omega_1 \wedge \omega_2 + t_1\omega_3 \wedge \omega_4 \\
dr &= -3rt_4\omega_1 + 3rt_3\omega_2 - rt_1\omega_3 + rt_2\omega_4 \\
ds &= -5st_1\omega_3 \\
dt_1 &= (4s^2 - t_1^2)\omega_3 \\
dt_2 &= m_1\omega_1 + m_2\omega_2 - t_1t_2\omega_3 + (t_1^2 + t_2^2 + s^2 - 9t_5^2)\omega_4 \\
dt_3 &= m_3\omega_1 + (m_4 - 2r^2 + t_1^2 + t_2^2 + t_3^2 + t_4^2 + 15t_5^2 + s^2)\omega_2 - t_1t_3\omega_3 \\
&\qquad + \left(t_2t_3 - 2t_4t_5 + \frac{1}{3}m_2\right)\omega_4 \\
dt_4 &= m_4\omega_1 - (m_3 + 2t_2t_5)\omega_2 - t_1t_4\omega_3 + \left(2t_3t_5 + t_2t_4 - \frac{1}{3}m_1\right)\omega_4 \\
dt_5 &= \frac{1}{3}m_2\omega_1 - \frac{1}{3}m_1\omega_2 - t_1t_5\omega_3 + 2t_2t_5\omega_4
\end{aligned}$$

for some functions m_1, m_2, m_3, m_4. Differentiation of these equations does not lead to new relations among the quantities. The differential ideal on the manifold $M = P_L \times \mathbb{R}^3$ is involutive, since the Cartan characters can be computed as $s_1 = 4, s_2 = s_3 = s_4 = 0$ and the space of integral elements at each point is parameterized by the 4 parameters m_1, m_2, m_3, m_4.

The structure equations imply $d(s^{\frac{8}{5}} + t_1^2 s^{-\frac{2}{5}}) = 0$. Since F and L are connected, it follows that there exists a constant $c > 0$ so that $s^{\frac{8}{5}} + t_1^2 s^{-\frac{2}{5}} = c^{\frac{8}{5}}$. Therefore, there is a function θ, well-defined on L, that satisfies:

$$s^{\frac{4}{5}} = c^{\frac{4}{5}} \cos 4\theta, \quad s^{-\frac{1}{5}} t_1 = c^{\frac{4}{5}} \sin 4\theta, \quad |\theta| < \frac{\pi}{8}.$$

From the sixth equation of (3.43), it follows that

$$\omega_3 = \frac{d\theta}{c(\cos 4\theta)^{\frac{5}{4}}}.$$

The structure equations imply that $\omega_1 = \omega_2 = \omega_4 = 0$ is integrable and also that $\omega_3 = 0$ defines an integrable 3-plane field on L. The 1-dimensional leaves of the field Γ_1 defined by $\omega_1 = \omega_2 = \omega_4 = 0$ are congruent along Γ_2, the codimension 1 foliation defined by $\omega_3 = 0$. This is clear since:

$$de_3 = -3s\omega_3 Je_3, \quad d(Je_3) = 3s\omega_3 e_3 \quad \mod \{\omega_1, \omega_2, \omega_4\}$$

and $ds = 0 \mod \omega_3$, meaning s is constant along each leaf of Γ_2. The above equations imply that the leaves of the Γ_1 foliation are congruent plane curves of curvature $-3s$, lying in the complex line (e_3, Je_3).

The form of the structure equations tells us that these examples must be related to the asymptotically conical special Lagrangian submanifolds, as seen in [1] for the $\mathbb{Z}_3$-stabilizer type case of the special Lagrangian 3-folds.

Suppose that the plane curves which are the leaves of the Γ_1 foliation are of the form $\operatorname{Re} z^{\frac{1}{p}} = c^{\frac{1}{p}}$, where c is a constant and $p \in \mathbb{R}$ is to be determined. By dilation, we can take $c = 1$ and consider the curve given by $z(t) = (1 + it)^p$, in the (e_3, Je_3)-plane. To compute the curvature of this curve, we use the formula for the curvature in any parametrization and we get:

$$k(t) = \frac{z' \wedge z''}{(\frac{d\tilde{s}}{dt})^3 e_3 \wedge Je_3} = \frac{p-1}{p}(1 + t^2)^{-\frac{p+1}{2}}. \tag{3.44}$$

Since $k = -3s$, and also using the sixth and the seventh structure equations in (3.43), we compute that $p = \frac{1}{4}$.

Therefore, the leaves of the Γ_1-foliation are curves given by the equation $\operatorname{Re} z^4 = c$, where c is a constant. From the equation of the curvature (3.44), it follows that as $t \to \infty$, $k \to 0$, so these curves flatten out, telling us that they have an asymptote.

Now we study the Γ_2-foliation, whose leaves are 3-manifolds. If we set $\theta = 0$, i.e., $t_1 = 0$ and $s = c$, we obtain a 3-manifold Σ, immersed in the 7-sphere S^7. This is clear since:

$$d(Je_3) = -s\omega_1 e_1 - s\omega_2 e_2 - s\omega_4 e_4 = -s dx,$$

where $x\colon \Sigma \to \mathbb{C}^4$ is the position vector. Since s is constant on Σ, it implies that

$$Je_3 = -sx + \text{ constant},$$

where we can suppose, by translation, that the constant is 0. Therefore, $x = -\frac{Je_3}{s}$ and Σ is immersed in the 7-sphere of radius $\frac{1}{s}$, in the direction Je_3.

The structure equations of the leaves of the $\omega_3 = 0$ foliation are:

$$\begin{aligned} d\omega_1 &= -t_2\omega_4 \wedge \omega_1 - 2t_5\omega_4 \wedge \omega_2 + t_3\omega_1 \wedge \omega_2 \\ d\omega_2 &= t_4\omega_1 \wedge \omega_2 + t_2\omega_2 \wedge \omega_4 + 2t_5\omega_4 \wedge \omega_1 \qquad \mod \omega_3 \\ d\omega_4 &= 6t_5\omega_1 \wedge \omega_2. \end{aligned}$$

Consider now the following expressions:

$$\begin{aligned} \eta_i &= s^{\frac{1}{5}}\omega_1,\ i = 1,2,4, \\ q_i &= s^{-\frac{1}{5}}t_i,\ i = 2,\dots,5, \\ p &= s^{-\frac{1}{5}}r, \\ v_i &= s^{-\frac{2}{5}}m_i,\ i = 1,\dots,4. \end{aligned}$$

The structure equations derived earlier show that

$$\begin{aligned} d\eta_1 &= q_2\eta_1 \wedge \eta_4 + q_3\eta_1 \wedge \eta_2 + 2q_5\eta_2 \wedge \eta_4 \\ d\eta_2 &= q_2\eta_2 \wedge \eta_4 + q_4\eta_1 \wedge \eta_2 + 2q_5\eta_4 \wedge \eta_1 \\ d\eta_4 &= 6q_5\eta_1 \wedge \eta_2 \\ dq_2 &= v_1\eta_1 + v_2\eta_2 + \left(\frac{4}{9} + q_2^2 - 9q_5^2\right)\eta_4 \end{aligned} \tag{3.45}$$

$$\begin{aligned}
dq_3 &= v_3\eta_1 + \left(\frac{4}{9} + q_2^2 + q_3^2 + q_4^2 + 15q_5^2 - 2p^2 + v_4\right)\eta_2 \\
&\quad + \frac{1}{3}(v_2 - 6q_4q_5 + 3q_2q_3)\eta_4 \\
dq_4 &= v_4\eta_1 - (v_3 + 2q_2q_5)\eta_2 - \frac{1}{3}(v_1 - 6q_3q_5 - 3q_2q_4)\eta_4 \\
dq_5 &= \frac{1}{3}v_2\eta_1 - \frac{1}{3}v_1\eta_2 + 2q_2q_5\eta_4 \\
dp &= -p(3q_4\eta_1 - 3q_3\eta_2 - q_2\eta_4).
\end{aligned}$$

Therefore, the metric $g = \eta_1^2 + \eta_2^2 + \eta_4^2$ is well-defined on each leaf of the Γ_2-foliation. The θ-curves meet the 3-manifold Σ orthogonally, so it is easy to see that the image of $(-\frac{\pi}{8}, \frac{\pi}{8}) \times \Sigma$ is of the form:

$$L_\Sigma = \{z\mathbf{u} \mid \mathbf{u} \in \Sigma,\ z \in \mathbb{C},\ \operatorname{Re} z^4 = c\}, \tag{3.46}$$

where c is a real constant. In order for this to be a special Lagrangian 4-fold, the cone on the image of Σ should be a special Lagrangian 4-fold.

We shall show now that, indeed, the cone on Σ is special Lagrangian with phase i. The cone on Σ is parameterized by:

$$(r, z) \to rz,\ r \in \mathbb{R}^+,\ z \in \Sigma^3.$$

The tangent space to $C(\Sigma)$ has a basis formed by the vectors:

$$\left(e_1 = \frac{\partial}{\partial x_1},\ e_2 = \frac{\partial}{\partial x_2},\ e_4 = \frac{\partial}{\partial x_4},\ Je_3 = \frac{\partial}{\partial y_3}\right).$$

Since

$$\omega = dx_1 \wedge dy_1 + dx_2 \wedge dy_2 + dx_3 \wedge dy_3 + dx_4 \wedge dy_4,$$

it is clear that $\omega \mid_{C(\Sigma)} = 0$, so the cone is Lagrangian. Also, $\Omega = dz_1 \wedge dz_2 \wedge dz_3 \wedge dz_4$ and we can easily compute that

$$\operatorname{Im} \Omega \mid_{C(\Sigma)} = dx_1 \wedge dx_2 \wedge dy_3 \wedge dx_4,$$

which represents the volume form on the cone, and $\mathrm{R}e\ \Omega \mid_{C(\Sigma)} = 0$. Therefore, $C(\Sigma)$ is special Lagrangian with phase i. Then, it is well-known [12] that (3.46) is a special Lagrangian 4-fold.

Theorem 3.21. *Let L be a connected special Lagrangian submanifolds in $\mathbb{C}^4$ such that its fundamental cubic at each point has a G-symmetry, where G is the order 18 normal subgroup of $\mathbf{D}_3 \times \mathbf{D}_3$. Then L is congruent to the product of two holomorphic curves in $\mathbb{C}^2$.*

Proof. Let L be a special Lagrangian 4-fold that satisfies the hypotheses of the theorem and let C be its fundamental cubic. Then

$$C = r(\omega_1^3 - 3\omega_1\omega_2^2) + v(\omega_3^3 - 3\omega_3\omega_4^2),$$

with $r > 0, v > 0, r \neq v$ defines a G-subbundle $F \subset P_L$ of the adapted coframe bundle $P_L \to L$, where G is the order 18 normal subgroup of $\mathbf{D}_3 \times \mathbf{D}_3$.

The structure equations on the subbundle F are computed to be:

$$\begin{aligned} d\omega_1 &= t_3\omega_1 \wedge \omega_2,\ d\omega_2 = t_4\omega_1 \wedge \omega_2 \\ d\omega_3 &= t_1\omega_3 \wedge \omega_4,\ d\omega_4 = t_2\omega_3 \wedge \omega_4 \\ dr &= -3rt_4\omega_1 + 3rt_3\omega_2 \\ dv &= -3vt_2\omega_3 + 3vt_1\omega_4 \\ dt_1 &= u_1\omega_3 + (t_1^2 + t_2^2 - 2v^2 + u_2)\omega_4 \\ dt_2 &= u_2\omega_3 - u_1\omega_4 \\ dt_3 &= u_3\omega_1 + (t_3^2 + t_4^2 - 2r^2 + u_4)\omega_2 \\ dt_4 &= u_4\omega_1 - u_3\omega_2 \end{aligned} \tag{3.47}$$

for some functions u_1, u_2, u_3, u_4.

From the above structure equations, we can see that $\omega_1 = \omega_2 = 0$ and $\omega_3 = \omega_4 = 0$ define integrable 2-plane fields on L. The structure equations also show that the leaves of the 2-plane field Γ_1 defined by $\omega_3 = \omega_4 = 0$ are congruent along Γ_2, the codimension 2 foliation defined by $\omega_1 = \omega_2 = 0$. Also, the 2-dimensional leaves of the 2-plane field Γ_2 are congruent along Γ_1.

Since $d(e_1\omega_1 + e_2\omega_2) = 0$ and $d(e_3\omega_3 + e_4\omega_4) = 0$, it follows that $e_1\omega_1 + e_2\omega_2 = d\pi_1$ and $e_3\omega_3 + e_4\omega_4 = d\pi_2$, where the projections $\pi_1 \colon L \to \Sigma_1$ and $\pi_2 \colon L \to \Sigma_2$ are well-defined. Therefore, $x = \pi_1 + \pi_2 + \mathrm{const}$ and L is the sum of two surfaces: $L = \Sigma_1 \times \Sigma_2$.

Since $d(e_1 \wedge e_2 \wedge Je_1 \wedge Je_2) = 0$, it follows that Σ_1 lies in the complex plane (e_1, e_2, Je_1, Je_2). Also, Σ_2 lies in the complex plane (e_3, e_4, Je_3, Je_4).

Because L is special Lagrangian, both surfaces Σ_1 and Σ_2 should be special Lagrangian 2-folds in $\mathbb{C}^2$. It is well-known then that these surfaces should be holomorphic curves with respect to some complex structure on $\mathbb{C}^2$. More explicitly, if $\Sigma_1 \subset \mathbb{C}^2$, with complex coordinates $\{z_1 = x_1 + iy_1, z_2 = x_2 + iy_2\}$, then Σ_1 is a holomorphic curve with respect to the complex coordinates $\{u_1 = x_1 - ix_2, v_1 = y_1 + iy_2\}$.

If $\Sigma_2 \subset \mathbb{C}^2$, with standard complex coordinates $\{z_3 = x_3 + iy_3, z_4 = x_4 + iy_4\}$, then Σ_2 is a holomorphic curve with respect to the complex coordinates $\{u_2 = x_3 - ix_4, v_2 = y_3 + iy_4\}$. q.e.d.

In the next case of $\mathbb{Z}_3$-symmetry we were unable to describe completely the SL 4-folds and therefore we have only a partial result.

Proposition 3.22. *There is an infinite parameter family of connected special Lagrangian submanifolds in $\mathbb{C}^4$ such that the fundamental cubic at each point has a $\mathbf{Z}_3$-symmetry and is of the form $(*)$. The family depends on 4 functions of one variable and the elements of this family are foliated by non-congruent minimal Legendrian surfaces in the direction $\{\omega_1, \omega_2\}$ and by congruent holomorphic curves in the direction $\{\omega_3, \omega_4\}$.*

Proof. Let L be a special Lagrangian 4-fold that satisfies the hypotheses of the theorem and let C be its fundamental cubic. Then

$$C = r(\omega_1^3 - 3\omega_1\omega_2^2) + 3s(\omega_3^2 + \omega_4^2 - 2\omega_1^2 - 2\omega_2^2)\omega_3 \\ + u(\omega_3^3 - 3\omega_3\omega_4^2) + v(\omega_4^3 - 3\omega_3^2\omega_4)$$

with $r > 0$ defines a $\mathbf{Z_3}$-subbundle $F \subset P_L$ of the adapted coframe bundle $P_L \to L$. The differential analysis shows that the structure equations on the bundle F are:

(3.48)

$$\begin{aligned}
&d\omega_1 = t_3\omega_1 \wedge \omega_2,\ d\omega_2 = t_4\omega_1 \wedge \omega_2,\ d\omega_3 = t_1\omega_3 \wedge \omega_4,\ d\omega_4 = t_2\omega_3 \wedge \omega_4,\\
&dr = -3rt_4\omega_1 + 3rt_3\omega_2 + 2rst_2\omega_3 + 2rst_1\omega_4\\
&ds = [-svt_1 + s(7s+u)t_2]\omega_3 + s[(3s-u)t_1 - vt_2]\omega_4\\
&du = [-11svt_1 + (3v^2 + 3u^2 + 6s^2 - su)t_2 + t_6]\omega_3\\
&\qquad + [(11su - 3u^2 - 6s^2 - 3v^2)t_1 - svt_2 - t_5]\omega_4\\
&dv = t_5\omega_3 + t_6\omega_4\\
&dt_1 = [v(t_1^2 + t_2^2 + 1) - 8st_1t_2]\omega_3 + [(u-s)(t_1^2 + t_2^2 + 1)]\omega_4\\
&dt_2 = -[(u+5s)(t_1^2 + t_2^2 + 1) - 8st_1^2]\omega_3 + v(t_1^2 + t_2^2 + 1)\omega_4\\
&dt_3 = -m_2\omega_1 + [4s^2(t_1^2 + t_2^2 + 1) + t_3^2 + t_4^2 - 2r^2 + m_1)\omega_2\\
&\qquad + 2st_2t_3\omega_3 + 2st_1t_3\omega_4\\
&dt_4 = m_1\omega_1 + m_2\omega_2 + 2st_2t_4\omega_3 + 2st_1t_4\omega_4
\end{aligned}$$

$$\begin{aligned}
dt_5 = \big[& -m_4 + 3t_1^2(30suv - 47s^2v - 3u^2v - 3v^3) \\
& + 3t_2^2(-7s^2v + 10suv - 3v^3 - 3u^2v) \\
& + (25s - 7u)t_1t_6 + 60sv^2t_1t_2 + (7u - 3s)t_2t_5 \\
& - 7vt_2t_6 - 7vt_1t_5 - 18s^2v - 6u^2v + 24suv - 6v^3\big]\omega_3 \\
& + [m_3 + (3s - u)t_1t_5 - vt_2t_5 + (s - u)t_2t_6 + vt_1t_6]\omega_4 \\
dt_6 = {} & m_3\omega_3 + m_4\omega_4
\end{aligned}$$

for some functions m_1, m_2, m_3, m_4.

The Cartan-Kähler analysis tells us that the solution should depend on 4 functions of 1 variable. From the structure equations, we can see that $\omega_3 = \omega_4 = 0$ and $\omega_1 = \omega_2 = 0$ define integrable 2-plane fields on L. Let Γ_1 be the $\omega_1 = \omega_2 = 0$ foliation and Γ_2 be the $\omega_3 = \omega_4 = 0$ foliation. The structure equations of the foliation Γ_1 show that the leaves are congruent and that the metric $g_1 = \omega_3^2 + \omega_4^2$ is well-defined on the leaf space of the Γ_1 foliation. It is easy to see that the leaves of the Γ_2 foliation are non-congruent.

Notice that if we denote $\Delta^2 = 4s^2(t_1^2 + t_2^2 + 1)$, then we get that $\frac{d\Delta}{\Delta} = 2s(t_2\omega_3 + t_1\omega_4)$. We see that Δ is constant on each leaf of the Γ_2 foliation. We compute that:

$$\begin{aligned}
d(\Delta\omega_1) &= t_3\omega_1 \wedge (\Delta\omega_2) = (t_3\omega_1 + t_4\omega_2) \wedge (\Delta\omega_2) \\
d(\Delta\omega_2) &= -t_4\omega_2 \wedge (\Delta\omega_1) = -(t_3\omega_1 + t_4\omega_2) \wedge (\Delta\omega_1)
\end{aligned} \tag{3.49}$$

and the metric $g_2 = (\Delta\omega_1)^2 + (\Delta\omega_2)^2$ is well-defined on the leaf space of the Γ_2 foliation.

Computations also show that $d(r^{\frac{1}{3}}\Delta^{\frac{2}{3}}\omega_1) = 0$ and $d(r^{\frac{1}{3}}\Delta^{\frac{2}{3}}\omega_2) = 0$. These imply that there are functions x_1, x_2 on L such that $r^{\frac{1}{3}}\Delta^{\frac{2}{3}}\omega_1 = dx_1$ and $r^{\frac{1}{3}}\Delta^{\frac{2}{3}}\omega_2 = dx_2$. This gives x_1, x_2 up to additive constants and $x_1 + ix_2$ is holomorphic with respect to the complex structure that $\{\omega_1, \omega_2\}$ define on the leaf space of Γ_2.

The above imply that the metric

$$g_2 = \left(\frac{\Delta}{r}\right)^{\frac{2}{3}}(dx_1^2 + dx_2^2) = F(x_1, x_2)(dx_1^2 + dx_2^2),$$

on the Γ_2 leaf space has Gauss curvature:

$$k = 1 - 2\left(\frac{r}{\Delta}\right)^2 = 1 - \frac{2}{F^3(x_1, x_2)}.$$

We obtain a differential equation for the function $F(x_1, x_2)$, given by:

$$\frac{1}{2}\Delta(\ln F) = \frac{2}{F^2} - F. \tag{3.50}$$

We used here the fact that a metric $ds^2 = e^{2u}(dx^2 + dy^2)$ is computed to have the Gauss curvature $K = -\Delta u e^{-2u}$. In our case, take $u = \frac{1}{2}\ln F$ and (3.50) follows. The function u satisfies the differential equation $\Delta u = e^{2u} - 2e^{-4u}$ (Tzitzeica equation), which is completely integrable by means of inverse scattering method [16]. This is the differential equation satisfied by the curvature of the metric of a minimal Legendrian immersion in $S^5(1)$, invariant under S^1-action, as shown by Mark Haskin in [4], p. 14. Sharipov [16] shows that the minimal immersion satisfying Tzitzeica equation are minimal tori which are complexly normal in S^5. Therefore, L is foliated by non-congruent minimal Legendrian surfaces in the direction $\{\omega_1, \omega_2\}$ and by congruent holomorphic curves in the direction $\{\omega_3, \omega_4\}$. We do not have a complete description of the family yet.

We move now to analyze the other orbit that stabilizes an element of order 3.

Case 2. $(r, s) = (\frac{2}{3}, \frac{1}{3})$: This case is equivalent to the $(r, s) = (\frac{1}{3}, \frac{1}{3})$ case, when the element in the maximal torus that stabilizes C is

$$g = \begin{pmatrix} e^{\frac{2\pi i}{3}} & 0 \\ 0 & e^{\frac{2\pi i}{3}} \end{pmatrix},$$

of **order 3**. The general harmonic cubic fixed by this element is

$$C = \mathrm{Re}(a_3 z_1^3 + 3a_2 z_1^2 z_2 + 3a_1 z_1 z_2^2 + a_0 z_2^3),\ a_0, a_1, a_2, a_3 \in \mathbb{C}.$$

We notice that the commutator of g is larger than the maximal torus in this case. The unitary group $\mathrm{U}(2)$ commutes with g and therefore we can use also its action to get rid of certain parameters, more precisely to make $a_0 = 0$ and $a_1, a_2 \in \mathbb{R}$. So, the general cubic stabilized by g will look like:

$$\begin{aligned}(**)\quad C = {} & u(x_1^3 - 3x_1x_2^2) + v(3x_1^2x_2 - x_2^3) \\ & + r[(x_1^2 - x_2^2)x_3 - 2x_1x_2x_4] + s[(x_3^2 - x_4^2)x_1 - 2x_2x_3x_4],\end{aligned}$$

where $u, v, r, s \in \mathbb{R}$.

Lemma 3.23. *The full stabilizer of the polynomial given by* ($**$), *where* $r, s, u, v \in \mathbb{R}$ *is:*

1) *a continuous subgroup of* $\mathrm{SO}(4)$, *if* $r = s = 0$ *or* $u = v = r = 0$ *or* $u = v = s = 0$;

2) *the dihedral subgroup* $\mathbf{D}_6$ *generated by the order* 6 *element* $\mathbf{a} = \begin{pmatrix} e^{\frac{4\pi i}{3}} & 0 \\ 0 & e^{\frac{\pi i}{3}} \end{pmatrix}$ *and the element of order* 2 *that flips the signs of* $\{x_2, x_4\}$, *if* $r = v = 0$;

3) *the dihedral subgroup* $\mathbf{D_3}$ *generated by the order* 3 *element* $\mathbf{g}$ *and the order* 2 *element that flips the signs of* $\{x_2, x_4\}$, *if* $s = v = 0$;

4) *the cyclic subgroup* $\mathbb{Z}_\mathbf{3}$ *generated by the order* 3 *element* $\mathbf{g}$ *if none of the above relations among the parameters* r, s, u, v *hold.*

Proof. We denoted by G be the stabilizer of the polynomial C. A simple computation shows that G is a continuous subgroup if and only if $r = s = 0$ or $u = v = r = 0$ or $u = v = s = 0$.

Doing the differential analysis in the discrete case, we obtain the following cases where the stabilizer becomes larger than $\mathbb{Z}_3$:

i) $r=0$. By making a rotation, if necessary, of angle $\theta = \frac{1}{3}\arctan(-\frac{v}{u})$ in the (x_1, x_2)-plane and of angle -2θ in the (x_3, x_4)-plane, we can suppose that $v = 0$ also.

The stabilizer of C for $r = v = 0$ is seen to be the dihedral subgroup $\mathbf{D}_6$ generated by the order 6 element $\mathbf{a} = \begin{pmatrix} e^{\frac{4\pi i}{3}} & 0 \\ 0 & e^{\frac{\pi i}{3}} \end{pmatrix}$ and the element of order 2 that flips the signs of $\{x_2, x_4\}$. This case of symmetry at least $\mathbb{Z}_6$ was already studied in Section 3.3.2 and it did not yield any families of special Lagrangian 4-folds.

ii) $s = 0$. In this case, we can arrange that $v = 0$ also and the stabilizer is computed to be the dihedral group $\mathbf{D_3}$.

iii) In the general case, when none of the above relations among the parameters r, s, u, v hold, the stabilizer of C is computed to be $\mathbb{Z}_3$ generated by the element g. q.e.d.

Theorem 3.24. *Let* L *be a connected special Lagrangian submanifold in* $\mathbb{C}^4$ *such that its fundamental cubic at each point has a* $\mathbf{Z}_3$-*symmetry and it is of the form* ($**$). *Then* L *is an* I-*special Lagrangian* J-*holomorphic surface in* $\mathbb{C}^4$, *where* $\{I, J, K\}$ *is the hyper-Kähler stucture on* $\mathbb{C}^4$.

Proof. Let L be a special Lagrangian 4-fold that satisfies the hypotheses of the theorem and let C be its fundamental cubic. From Lemma 3.23, the equation

$$\begin{aligned} C = {} & u(\omega_1^3 - 3\omega_1\omega_2^2) + v(3\omega_1^2\omega_2 - \omega_2^3) \\ & + r[(\omega_1^2 - \omega_2^2)\omega_3 - 2\omega_1\omega_2\omega_4]) + s[(\omega_3^2 - \omega_4^2)\omega_1 - 2\omega_2\omega_3\omega_4], \end{aligned}$$

with $r, s, u, v \in \mathbb{R}$ defines a $\mathbb{Z}_3$-subbundle $F \subset P_L$ of the adapted coframe bundle $P_L \to L$.

On the subbundle F, the following identities hold:

(3.51)

$$\begin{aligned} & (\beta_{ij}) \\ & = \begin{pmatrix} u\omega_1 + v\omega_2 + r\omega_3 & v\omega_1 - u\omega_2 - r\omega_4 & r\omega_1 + s\omega_3 & -r\omega_2 - s\omega_4 \\ v\omega_1 - u\omega_2 - r\omega_4 & -u\omega_1 - v\omega_2 - r\omega_3 & -r\omega_2 - s\omega_4 & -r\omega_1 - s\omega_3 \\ r\omega_1 + s\omega_3 & -r\omega_2 - s\omega_4 & s\omega_1 & -s\omega_2 \\ -r\omega_2 - s\omega_4 & -r\omega_1 - s\omega_3 & -s\omega_2 & -s\omega_1 \end{pmatrix}. \end{aligned}$$

The Cartan-Kähler analysis yields the following relations between the α_{ij}'s:

$$\alpha_{31} - \alpha_{42} = 0 \quad \text{and} \quad \alpha_{32} + \alpha_{41} = 0.$$

We consider the ideal I_1, on the coframe bundle, spanned by the 1-forms (3.51) and the two 1-forms $\alpha_{31} - \alpha_{42}$ and $\alpha_{32} + a_{41}$. The independence condition is given by $\omega_1 \wedge \omega_2 \wedge \omega_3 \wedge \omega_4 \neq 0$ and the tableau matrix for the structure equations is given by:

$$\begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\ \alpha_2 & -\alpha_1 & \alpha_4 & -\alpha_3 \\ \alpha_3 & \alpha_4 & \alpha_5 & \alpha_6 \\ \alpha_4 & -\alpha_3 & \alpha_6 & -\alpha_5 \\ \alpha_5 & \alpha_6 & -3s\pi_6 & -3s\pi_5 \\ \alpha_6 & -\alpha_5 & -3s\pi_5 & 3s\pi_6 \end{pmatrix}$$

where

$$\pi_1 = dr,\ \pi_2 = ds,\ \pi_3 = du,\ \pi_4 = dv,$$

$$\pi_5 = \alpha_{41},\ \pi_6 = \alpha_{42},\ \pi_7 = \alpha_{43},\ \pi_8 = \alpha_{21}$$

$$\alpha_1 = -\pi_3 + 3r\pi_6 + 3v\pi_8$$

$$\alpha_2 = -\pi_4 - 3r\pi_5 - 3u\pi_8$$
$$\alpha_3 = -\pi_1 + v\pi_5 + (2s-u)\pi_6$$
$$\alpha_4 = -(2s+u)\pi_5 - v\pi_6 - r\pi_7 - 2r\pi_8$$
$$\alpha_5 = -2r\pi_2 - \pi_6$$
$$\alpha_6 = -2r\pi_5 - 2s\pi_7 + s\pi_8.$$

From the above tableau, we compute the reduced Cartan characters as $s_1' = 6, s_2' = 2, s_3' = s_4' = 0$. The integral elements of the system at each point is shown to form a space of dimension $10 = s_1' + 2s_2' + 3s_3' + 4s_4'$ and therefore, by Cartan's Test, the system I_1 is involutive.

The form of the tableau resembles the tableau for the structure equations of a complex surface with complex structure given by $\gamma_1 = \omega_1 + i\omega_2$ and $\gamma_2 = \omega_3 + i\omega_4$. We will show that this is actually the case. The characteristic variety of the ideal is formed by 2 complex lines spanned by $\{\gamma_1, \gamma_2\}$ and their conjugates.

The first derived system of I_1 is generated by the rank 6 Pfaffian system I_2 spanned by the six 1-forms:

$$\begin{aligned} &\theta_1 = \beta_{11} + \beta_{22},\ \theta_2 = \beta_{33} + \beta_{44},\ \theta_3 = \beta_{41} - \beta_{32} \\ &\theta_4 = \beta_{31} + \beta_{42},\ \theta_5 = \alpha_{31} - \alpha_{42},\ \theta_6 = \alpha_{32} + \alpha_{41}. \end{aligned} \tag{3.52}$$

This system is Frobenius and defines a foliation of dimension 10 on the coframe bundle. The integral manifold of our original system will be a submanifold of the maximal integral manifold of the derived system I_2. Therefore, we will adapt frames and restrict to the first derived system, looking for integral manifolds of this system.

We notice that, when restricted to the first derived system, the connection matrix takes values in the Lie algebra of a 10-dimensional subgroup of SU(4). This subgroup can be shown to be $Sp(2)$. The system I_2 restricts to the $Sp(2)$-coframe bundle, of dimension 18. The canonical form on this bundle has components $\xi_i = \omega_i + i\eta_i$ and the 1-forms

$$\{\omega_i, \eta_i, \beta_{11}, \beta_{33}, \beta_{21}, \beta_{31}, \beta_{41}, \beta_{43}, \alpha_{21}, \alpha_{31}, \alpha_{41}, \alpha_{43}\}$$

form a basis for the space of 1-forms on this coframe bundle $P \cong \mathbb{C}^4 \times Sp(2)$. On the integral manifolds, $\eta_i = 0$ for $i = 1...4$.

Now, the symplectic group $Sp(2)$ leaves invariant 3 symplectic 2-forms $\{\zeta_1, \zeta_2, \zeta_3\}$. One of them is the Kähler form of the standard complex structure I on $\mathbb{R}^8$.

$$\begin{aligned}\zeta_1 &= \frac{i}{2}(\xi_1 \wedge \bar{\xi}_1 + \xi_2 \wedge \bar{\xi}_2 + \xi_3 \wedge \bar{\xi}_3 + \xi_4 \wedge \bar{\xi}_4) \\ &= \omega_1 \wedge \eta_1 + \omega_2 \wedge \eta_2 + \omega_3 \wedge \eta_3 + \omega_4 \wedge \eta_4\end{aligned}$$

and the other 2-forms are computed to be:

$$\begin{aligned}\zeta_2 &= \omega_1 \wedge \omega_2 + \omega_3 \wedge \omega_4 - \eta_1 \wedge \eta_2 - \eta_3 \wedge \eta_4 \\ \zeta_3 &= \omega_1 \wedge \eta_2 - \omega_2 \wedge \eta_1 + \omega_3 \wedge \eta_4 - \omega_4 \wedge \eta_3.\end{aligned}$$

Let I, J, K be the complex structures on $\mathbb{R}^8$ corresponding to left multiplication by the elementary quaternions i, j and k. Then the standard metric $g = \sum_{i=1}^{4}(\omega_i^2 + \eta_i^2)$ on $\mathbb{R}^8$ is Kähler with respect to each I, J, K, with Kähler form ζ_1, ζ_2 and ζ_3, respectively. The forms $\psi_1 = \zeta_2 + i\zeta_3,\ \psi_2 = \zeta_1 + i\zeta_3,\ \psi_3 = \zeta_1 + i\zeta_2$ are the holomorphic symplectic forms on $\mathbb{C}^4$, associated to the complex structures I, J and K respectively. The standard complex structure on $\mathbb{R}^8$ is considered to be I, given by the complex 1-forms: $\omega_j + i\eta_j,\ j = 1...4$. The 4-forms $\Omega_i = \frac{1}{2}\psi_i^2,\ i = 1...3$ are the holomorphic volume forms on $\mathbb{C}^4$, associated to the complex structures I, J and K, respectively, with Ω_1 being the usual holomorphic volume form. On the integral manifolds of I_2, $\zeta_1 = \zeta_3 = 0$ and $\zeta_2 = \omega_1 \wedge \omega_2 + \omega_3 \wedge \omega_4$ is the Kähler form for the complex structure J given by $\gamma_1 = \omega_1 + i\omega_2$ and $\gamma_2 = \omega_3 + i\omega_4$.

We will now show that the integral manifold of the ideal generated by the 2-forms ζ_1 and ζ_3 are complex manifolds with respect to the complex structure J. Let (z_1, z_2, z_3, z_4) be the complex coordinates on $\mathbb{R}^8$ that are holomorphic for the complex structure J. Then:

$$\zeta_1 + i\zeta_3 = dz_1 \wedge dz_2 + dz_3 \wedge dz_4$$

Let $\mathbf{I}$ be the differential ideal generated by the complex 1-form $\psi_2 = \zeta_1 + i\zeta_3$. We use the Cartan-Kähler analysis [1] to compute the Cartan characters as $s_1 = s_2 = 2$ and $s_3 = s_4 = 0$. The space of 2-dimensional integral elements over a point has dimension $6 = s_1 + 2s_2 + 3s_3 + 4s_4$ and by Cartan's Test, the system is involutive. The maximal integral manifolds of this ideal are given by 2 complex linear equations, i.e., they are J-holomorphic surfaces in $\mathbb{C}^4$.

An integral manifold of the derived system I_2 is an integral manifold of the system $\zeta_1 = \zeta_3 = 0$ and an integral manifold of the system $\zeta_1 = \zeta_3 = 0$ is an integral manifold of the derived system. To summarize, the integral manifolds Σ of our original system are J-holomorphic surfaces in $\mathbb{C}^4$. They are I-special Lagrangian 4-folds, because

$$\zeta_1 \mid_\Sigma = 0 \text{ and } \frac{1}{2}\operatorname{Im}(\Omega_1^2) \mid_\Sigma = \zeta_2 \wedge \zeta_3 \mid_\Sigma = 0.$$

q.e.d.

Theorem 3.25. *Let L be a connected special Lagrangian submanifold in $\mathbb{C}^4$ such that its fundamental cubic at each point has a $\mathbf{D}_3$-symmetry and it is of the form $(*)$ with $s = v = 0$. Then L is a ruled I-special Lagrangian J-holomorphic surface in $\mathbb{C}^4$.*

Proof. The analysis here is similar to the one in the previous result. It can be shown that the solutions are again I-special Lagrangian J-holomorphic surfaces. Moreover, the structure equations show that the holomorphic surfaces are foliated by planes in the $\{e_3, e_4\}$-direction. The conclusion is that the solutions are ruled I-special Lagrangian J-holomorphic surfaces. q.e.d.

We conclude this paper with the following:

Open Problem. It remains to study the general case when the symmetry of the fundamental cubic is at least a $\mathbb{Z}_2$. This is the most complicated case since the space of fixed harmonic cubics involves a large number of parameters.

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